

# The Fourier Transform and Related Topics

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## 1 Introduction

The Fourier transform is one of the key tools in solving and studying partial differential equations. Indeed, a deep understanding of the Fourier transform has created the careers of many highly respected mathematicians. In addition, understanding the Fourier transform paves the way for understanding other integral transforms (e.g. the Laplace, Radon, Mellin and X-Ray transforms). In the words of Prof. Michael Eastwood (University of Adelaide, Australia), “All good integral transforms are really just the Fourier transform.”

The aim of these notes is twofold: first to provide some of the theoretical background regarding the Fourier transform and PDE, and second to provide some intuition for how the Fourier transform works. I wrote these notes more or less based on lecture notes I took in courses taught by Ken Bube [1] and Gunther Uhlman [4].

The subject of the Fourier transform, and integral transforms in general, is vast and deep; these notes cannot do more more than scratch its surface. In particular, these notes will *not* treat convergence properties in a rigorous fashion. Many of the computations below are merely formal computations, and to justify them one must understand important questions, such as when one can interchange the order of integration. These questions are the territory of a graduate-level analysis course (or several such courses). Rather than deal with such sticky issues as Fubini’s theorem, Hölder’s inequality, etc., I will blithely differentiate underneath the integral sign, interchange integrals, and so on. Fortunately, there are many fabulous books about the Fourier transform. One of my favorite books is Folland’s book [2].

The organization of these notes is as follows: in section 2 discusses some basic properties of convolutions. Section 3 introduces the Fourier transform and collects some of its basic properties. Section 4 proves the Fourier inversion formula. Sections 5 and 6 use the Fourier transform to treat the wave and heat equations, respectively, on a line.

## 2 Convolutions

One can think of the convolution of two functions as an averaging of one function against the other. It is defined when  $f$  and/or  $g$  decay fast enough so that the improper integral converges. More precisely:

**Definition 1** *Given two functions  $f$  and  $g$ , we define the convolution of  $f$  and  $g$  by*

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)ds.$$

For instance, if  $f$  is bounded and  $\int_{-\infty}^{\infty} |g(y)|dy$  is finite then  $f * g$  is well defined (i.e. the improper integral converges). In this case

$$|f * g(x)| \leq \max |f| \int_{-\infty}^{\infty} |g(y)|dy.$$

(Exercise: derive this bound.)

## 2.1 Brief remarks on convergence

How fast does  $f$  need to decay so that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty?$$

One can gain some insight into this question by looking at simple examples. For instance, suppose  $f$  behaves like  $|x|^p$  for some power  $p$  and for large  $x$ . Then the total integral  $\int |f|$  of  $f$  behaves like

$$\int_1^{\infty} x^p dx.$$

For  $p \neq 0, -1$  this last integral is

$$\int_1^{\infty} x^p dx = \frac{1}{p} x^{p+1} \Big|_1^{\infty},$$

which is finite precisely when  $p < -1$ . For  $p = 0$  the integral is infinite, and for  $p = -1$  you get a natural log term, which is again infinite. Thus we have the

**Proposition 1** *Let  $f = f(x)$  be a real-valued, piece-wise continuous function on the real line. If there are numbers  $c > 0$  and  $p < -1$  such that*

$$|f(x)| \leq c(1 + |x|)^p$$

*then  $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ .*

## 2.2 Some examples and properties

Suppose  $f = 1$  and  $\int_{-\infty}^{\infty} |g(y)| dy < \infty$ . Then the convolution

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) ds = \int_{-\infty}^{\infty} g(y) dy$$

is the constant function  $\int g$ .

Suppose  $f(x) = x$  and  $g$  satisfies  $\int |g| < \infty$  and  $\int |g'| < \infty$ . Then

$$\begin{aligned} f * g'(x) &= \int_{-\infty}^{\infty} f(x-y)g'(y) dy \\ &= \int_{-\infty}^{\infty} (x-y)g'(y) dy \\ &= x \int_{-\infty}^{\infty} g(y) dy - \int_{-\infty}^{\infty} yg'(y) dy \\ &= x \int_{-\infty}^{\infty} g'(y) dy + \int_{-\infty}^{\infty} g(y) dy. \end{aligned}$$

In the last step I integrated by parts. The boundary terms in the integration by part are both zero because  $g(y) \rightarrow 0$  as  $y \rightarrow \pm\infty$ . Notice that this time we get a linear function for  $f * g$ , regardless of what  $g$  is, so long as  $g$  decays fast enough. There is a general rule hiding here: the convolution of a polynomial  $f$  and any function  $g$  with enough decay is another polynomial, of the same order as  $f$ .

Next I will list some important features of convolutions. Firstly, the convolution of two functions does not depend on the order of the functions:

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(x-y)g(y)dy \\ &= \int_{-\infty}^{\infty} f(z)g(x-z)dz = g * f(x). \end{aligned}$$

Using the fact that  $f * g = g * f$ , one can take the derivative of  $f * g$  two different ways:

$$\frac{d}{dx}f * g(x) = \frac{d}{dx} \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} \frac{d}{dx}(f(x-y)g(y))dy = \int_{-\infty}^{\infty} f'(x-y)g(y)dy$$

or

$$\frac{d}{dx}f * g(x) = \frac{d}{dx}g * f(x) = \int_{-\infty}^{\infty} g'(x-y)f(y)dy.$$

(Exercise: prove that if  $f$  is a polynomial of order  $k$  and  $g$  is any function with sufficient decay, then  $f * g$  is again a polynomial of order  $k$ .)

One should be a bit careful with convolutions, as they have some unusual properties. For instance, if  $f(x) = \frac{1}{x+i}$  and  $g(x) = \frac{1}{x-i}$  then  $f * g(x) = 0$  for all  $x$ , even though neither  $f$  or  $g$  are ever zero!. The proof of this is computation involving the Fourier transform, which we may see a bit later. However, if  $f * f(x) = 0$  for all  $x$ , then  $f$  is really the zero function.

### 2.3 An approximate identity

Let  $\Phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$  and for  $\epsilon > 0$  define  $\Phi_\epsilon(x) = \frac{1}{\epsilon}\Phi(\frac{x}{\epsilon})$ . You may recognize  $\Phi$  as the Gaussian distribution from probability and statistics. It also appears frequently in quantum physics and statistical mechanics. Notice  $\Phi > 0$  and  $\Phi(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . In fact, we have the following

**Lemma 2** For all  $\epsilon > 0$ ,

$$\int_{-\infty}^{\infty} \Phi_\epsilon(x)dx = 1.$$

**Proof:** We begin with the observation that, using the change of variables  $y = x/\epsilon$ ,

$$\int_{-\infty}^{\infty} \Phi_\epsilon(x)dx = \int_{-\infty}^{\infty} \Phi(y)dy.$$

Thus it suffices to show that  $\int_{-\infty}^{\infty} \Phi(x)dx = 1$ . We have

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx\right)^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dydx \\ &= \int_0^{2\pi} \int_0^{\infty} re^{-\frac{1}{2}r^2} drd\theta \\ &= 2\pi \left(-e^{-\frac{1}{2}r^2}\right)\Big|_{r=0}^{r=\infty} = 2\pi. \end{aligned}$$

□

However, as  $\epsilon \rightarrow 0$ ,  $\Phi_\epsilon(x)$  is very small for  $|x| \geq \epsilon$ . In fact, everywhere except  $x = 0$ ,  $\Phi_\epsilon \rightarrow 0$  uniformly. The value of  $\Phi_\epsilon(0)$  grows arbitrarily large as  $\epsilon \rightarrow 0$ . In the limit, one obtains what is commonly called the Dirac delta function centered at  $x = 0$ , denoted  $\delta_0$ . A nice characterization of the Dirac delta function is the statement that

$$\int_{-\infty}^{\infty} \delta_0(x)f(x)dx = f(0). \tag{1}$$

Strictly speaking,  $\delta_0$  is not a function; rather it is a generalization of a function called a distribution (see [3]). However, you can just think of  $\delta_0$  as  $\Phi_\epsilon$  for  $\epsilon$  very small, that is, a rescaled Gaussian distribution.

One can recover equation (1) from the limit of integrals involving  $\Phi_\epsilon$  as  $\epsilon \rightarrow 0$ . Indeed,

$$\int_{-\infty}^{\infty} |f(x) - \Phi_\epsilon * f(x)| dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x) - \Phi_\epsilon(x-y)f(y)| dy dx \rightarrow 0$$

as  $\epsilon \rightarrow 0$ . This is because for  $y$  far away from  $x$ ,  $\Phi_\epsilon(x-y)$  is very small. This computation shows  $\Phi_\epsilon * f(x)$  is close to  $f(x)$  for all  $x$ , because if it were not then the integral above would be bounded away from zero. We can summarize these properties with the following

**Proposition 3** *Let  $f$  be a bounded, piece-wise continuous function. Then*

1. *for any  $\epsilon < 0$ ,  $\Phi_\epsilon * f$  is a bounded function with infinitely many derivatives and*
2.  *$\Phi_\epsilon * f \rightarrow f$  uniformly on bounded sets as  $\epsilon \rightarrow 0$ .*

For this reason,  $\Phi_\epsilon$  is called an *approximate identity* for  $\epsilon$  small. (It is also sometimes called a *mollifier*.) The reason for this terminology is that for  $\epsilon > 0$  small,  $\Phi_\epsilon * f$  is a function which is everywhere very close to  $f$ .

### 3 The Fourier Transform

One can think of the Fourier transform of a function  $f$  as smearing the function values of  $f$  over the whole real line. More precisely:

**Definition 2** *The Fourier transform of a function  $f$  is defined by*

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx.$$

Here  $i$  is the square root of  $-1$ .

Notice that the Fourier transform of  $f$  is the convolution of  $f$  against a complex exponential, which is highly oscillatory.

Again, the Fourier transform  $\hat{f}$  is defined only when the improper integral converges. For instance, if  $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ , then  $\hat{f}(\omega)$  exists for all  $\omega$  and

$$|\hat{f}(\omega)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| dx.$$

In fact, under this condition ( $\int |f| < \infty$ ),  $\hat{f}(\omega) \rightarrow 0$  as  $\omega \rightarrow \pm\infty$ . (Exercise: derive the bound listed above and prove that in this case  $\hat{f}(\omega) \rightarrow 0$  as  $\omega \rightarrow \pm\infty$ .)

#### 3.1 Some properties

Notice that the Fourier transform of any function  $f$  (provided  $f$  decays sufficiently so that the improper integral converges) has as many derivatives as you please: just differentiate the exponential under the integral sign. In particular,  $f$  need not even be continuous and  $\hat{f}$  can still have infinitely many derivatives.

The Fourier transform enjoys many nice properties, some of which are listed below.

First note that for an  $a \in \mathbb{R}$ ,

$$\mathcal{F}(e^{iax}f(x))(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} e^{iax} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(\omega-a)x} f(x) dx = \hat{f}(\omega - a).$$

Similarly,

$$\mathcal{F}(f(x-y))(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x-y) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(z+y)} f(z) dz = e^{-iy\omega} \hat{f}(\omega).$$

Here we have used the change of variables  $z = x - y$ . We can summarize these last two properties by saying that the Fourier transform intertwines translation and multiplication by a complex exponential.

Also, (provided  $f$  and  $f'$  decay quickly enough)

$$\mathcal{F}(f')(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f'(x) dx = \frac{1}{\sqrt{2\pi}} \left. -i\omega e^{-ix\omega} f(x) \right|_{x=-\infty}^{x=\infty} + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx = i\omega \hat{f}(\omega).$$

Here we have used the fact that  $f(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$  to conclude that the boundary terms in the integration by parts both go to zero. There is similarly a nice formula for  $\frac{d}{d\omega} \hat{f}(\omega)$ :

$$\begin{aligned} \frac{d}{d\omega} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \frac{d}{d\omega} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{d\omega} e^{-ix\omega} f(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} (-ix) f(x) dx = \mathcal{F}(-ixf)(\omega). \end{aligned}$$

In other words, the Fourier transform intertwines differentiation and multiplication by a polynomial. This property of the Fourier transform is perhaps the most useful property it has.

The Fourier transform of a convolution has a very pretty expression:

$$\begin{aligned} \mathcal{F}(f * g)(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f * g(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} \int_{-\infty}^{\infty} f(x-y) g(y) dy dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\omega} f(x-y) g(y) dx dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\omega} e^{-iy\omega} e^{iy\omega} f(x-y) g(y) dx dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(x-y)} f(x-y) d(x-y) \cdot \int_{-\infty}^{\infty} e^{-iy\omega} g(y) dy \\ &= \sqrt{2\pi} \hat{f}(\omega) \cdot \hat{g}(\omega). \end{aligned}$$

Thus the Fourier transform of a convolution is the product of the Fourier transforms.

It is also instructive to separate the Fourier transform into sine and cosine parts. Indeed,

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cos(x\omega) - i \sin(x\omega)) f(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(x\omega) f(x) dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(x\omega) f(x) dx. \end{aligned}$$

The first term in the final expression above is often called the Fourier cosine transform, and it is the real part of the Fourier transform. Meanwhile, the second term is often called the Fourier sine transform, and it is (up to a sign) the imaginary part of the Fourier transform. Notice that the Fourier sine transform of an even function is zero and (and therefore the Fourier transform of an even function is real-valued). (Exercise: formulate the corresponding statement for an odd function.)

### 3.2 Some examples

There are some Fourier transforms we can easily compute, which we will discuss in this section.

We will begin with a square pulse:

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

Then

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-ix\omega} dx \\ &= -\frac{1}{i\omega\sqrt{2\pi}} e^{-ix\omega} \Big|_{-1}^1 = -\frac{1}{i\omega\sqrt{2\pi}} (e^{-i\omega} - e^{i\omega}) \\ &= \frac{2 \sin \omega}{\sqrt{2\pi}\omega}. \end{aligned}$$

Now let  $f(x)$  be defined by

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0. \end{cases}$$

Then

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(1+i\omega)} dx = \frac{1}{\sqrt{2\pi}} - \frac{1}{1+i\omega} e^{-x(1+i\omega)} \Big|_{t=0}^{t=\infty} = \frac{1}{\sqrt{2\pi}(1+i\omega)}.$$

The reason we truncated the exponential in  $f$  was to ensure that  $f$  has sufficient decay so that the Fourier transform is well-defined. Recall that  $e^{-x} \rightarrow \infty$  as  $x \rightarrow -\infty$ , so if we let  $g(x) = e^{-x}$  then  $\int_{-\infty}^{\infty} e^{-ix\omega} g(x) dx = \int_{-\infty}^{\infty} e^{-ix\omega} e^{-x} dx$  is a divergent improper integral.

We can also compute the Fourier transform of  $xf(x)$ :

$$\mathcal{F}(xf)(\omega) = i \frac{d}{d\omega} \hat{f}(\omega) = i \frac{1}{\sqrt{2\pi}} \frac{d}{d\omega} \left( \frac{1}{1+i\omega} \right) = \frac{1}{\sqrt{2\pi}(1+i\omega)^2}.$$

Again, to do this computation rigorously, one must justify that  $xf(x)$  decays fast enough so that its Fourier transform is well defined.

Recall that we defined  $\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$  in section 2. To find the Fourier inversion formula we will need the following lemma.

**Lemma 4**  $\hat{\hat{\Phi}} = \Phi$ .

**Proof:** First notice that

$$\frac{d}{dx} \Phi = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2} = -x\Phi.$$

The taking the Fourier transform, we see

$$-i \frac{d}{d\omega} \hat{\Phi} = \mathcal{F}(-x\Phi) = \mathcal{F}(\Phi') = i\omega \hat{\Phi},$$

which we can rewrite as

$$\frac{d}{d\omega} \hat{\Phi} = -\omega \hat{\Phi}.$$

Solving this differential equation for  $\hat{\Phi}$ , we see

$$\hat{\Phi}(\omega) = \hat{\Phi}(0) e^{-\omega^2/2},$$

so it remains only to find  $\hat{\Phi}(0)$ . However,

$$\hat{\Phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(x) dx = \frac{1}{\sqrt{2\pi}},$$

so

$$\hat{\Phi}(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} = \Phi(\omega).$$

□

Notice that by the change of variables  $y = \frac{x}{\epsilon}$ , we have

$$\hat{\Phi}_\epsilon(\omega) = \hat{\Phi}(\epsilon\omega) = \Phi(\epsilon\omega) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon^2\omega^2/2}. \quad (2)$$

The above lemma also shows

$$\begin{aligned} \Phi_\epsilon(x) &= \frac{1}{\epsilon} \Phi\left(\frac{x}{\epsilon}\right) = \frac{1}{\epsilon} \hat{\Phi}\left(\frac{x}{\epsilon}\right) = \frac{1}{\epsilon} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\frac{x}{\epsilon}\omega} \Phi(\omega) d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} \Phi(\epsilon y) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixz} \Phi(\epsilon z) dz. \end{aligned} \quad (3)$$

Here we have used two changes of variables: first  $y = \frac{x}{\epsilon}$  and then  $z = -y$ . We have also used  $\Phi(y) = \Phi(-y)$  (i.e.,  $\Phi$  is an even function). We will use equation (4) to derive the Fourier inversion formula.

## 4 The Fourier inversion formula

In this section we will derive a formula for the inverse Fourier transform  $\mathcal{F}^{-1}(f) = \check{f}(x)$ .

**Theorem 5**  $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} \hat{f}(\omega) d\omega$ .

**Proof:** We will use equation (4):

$$\Phi_\epsilon(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixz} \Phi(\epsilon z) dz.$$

Then

$$\begin{aligned} f * \Phi_\epsilon(x) &= \int_{-\infty}^{\infty} f(x-y) \Phi_\epsilon(y) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) e^{iy\omega} \Phi(\epsilon\omega) d\omega dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) e^{iy\omega} e^{-ix\omega} e^{ix\omega} \Phi(\epsilon\omega) d\omega dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) e^{-i\omega(x-y)} e^{ix\omega} \Phi(\epsilon\omega) d\omega dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} e^{-i\omega(x-y)} f(x-y) d(x-y) \right] e^{ix\omega} \Phi(\epsilon\omega) d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{ix\omega} \Phi(\epsilon\omega) d\omega \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} \hat{f}(\omega) d\omega \end{aligned}$$

However, we know that

$$f * \Phi_\epsilon(x) \rightarrow f(x)$$

as  $\epsilon \rightarrow 0$ , so the above computation shows

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} \hat{f}(\omega) d\omega.$$

□

This yields a formula for the inverse transform of the Fourier transform:

$$\mathcal{F}^{-1}(g)(x) = \check{g}(x) = \int_{-\infty}^{\infty} e^{ix\omega} g(\omega) d\omega.$$

The above formula is known as the Fourier inversion formula.

You may recall that back in section 2 I listed two nowhere zero functions  $f(x) = \frac{1}{x+i}$  and  $g(x) = \frac{1}{x-i}$  such that the convolution  $f * g$  is everywhere zero. Now we can finally prove this remarkable property. If we define  $h$  by

$$h(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$

we have already seen that  $\hat{h}(\omega) = -i\frac{1}{\omega-i}$ . Also,  $\hat{h}(-\omega) = i\frac{1}{\omega+i}$ . This implies

$$f(\omega) = \frac{1}{\omega-i} = i\hat{h}(\omega) \quad g(\omega) = \frac{1}{\omega+i} = -i\hat{h}(-\omega).$$

Now we are ready to compute the convolution.

$$f * g(t) = (\mathcal{F}(f(\omega)) * \mathcal{F}(f(-\omega)))(t) = \mathcal{F}(f(\omega) \cdot f(-\omega)) = 0$$

because for all  $\omega$  either  $f(\omega) = 0$  or  $f(-\omega) = 0$  (and the Fourier transform of zero is zero).

## 5 The wave equation

There are several methods for solving the wave equation on a line. First, we must be more precise about what equation we are solving. In this section we will consider the following initial value problem:

$$\begin{cases} \partial_t^2 u(x, t) &= \partial_x^2 u(x, t) \\ u(x, 0) &= f(x) \\ \partial_t u(x, 0) &= g(x). \end{cases} \quad (4)$$

We will take  $u = u(x, t)$  to be defined for all  $x$  and  $t$ . For technical reasons, we will need some decay conditions on the  $f$  and  $g$ , so that we can apply the Fourier transform and inverse Fourier transform. For instance, we might want both  $f(x)$  and  $g(x)$  to be zero for  $|x|$  sufficiently large; this condition is, of course, sufficient to make sense of the Fourier transform, but it is not necessary. Or we might suppose that  $f(x)$  and  $g(x)$  decay very quickly as  $x \rightarrow \pm\infty$ . One can think of these decay conditions as boundary conditions on  $u$  as  $x \rightarrow \pm\infty$ .

The key idea in solving (4) is to apply the Fourier transform in the  $x$ -variable *only*. If we denote this transform as

$$\hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} u(x, t) dx,$$

the differential equation becomes

$$\hat{u}'' = -\omega^2 \hat{u},$$

with the initial conditions

$$\hat{u}(\omega, 0) = \hat{f}(\omega), \quad \hat{u}'(\omega, 0) = \hat{g}(\omega).$$

Here the prime denotes differentiation with respect to  $t$ . This is now just an ODE for  $\hat{u}$  in the  $t$  variable, with the parameter  $\omega$ . We have a second order ODE

$$\hat{u}'' + \omega^2 \hat{u},$$

which has solutions

$$\hat{u}(\omega, t) = c_1(\omega) \cos(\omega t) + c_2(\omega) \sin(\omega t).$$

Notice that

$$\hat{u}(\omega, 0) = c_1(\omega), \quad \hat{u}'(\omega, 0) = \omega c_2(\omega).$$

Next we need to match the initial conditions:

$$\hat{f}(\omega) = \hat{u}(\omega, 0) = c_1(\omega)$$

and

$$\hat{g}(\omega) = \hat{u}'(\omega, 0) = \omega c_2(\omega).$$

Thus the solution to the transformed problem (for  $\hat{u}$ ) is

$$\hat{u}(\omega, t) = \hat{f}(\omega) \cos(\omega t) + \frac{\hat{g}(\omega)}{\omega} \sin(\omega t).$$

We can now find  $u = u(x, t)$  by doing the inverse Fourier transform:

$$u(x, t) = \mathcal{F}^{-1}(\hat{f} \cos(\omega t) + \frac{\hat{g}}{\omega} \sin(\omega t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} \hat{f}(\omega) \cos(\omega t) d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} \frac{\hat{g}(\omega)}{\omega} \sin(\omega t) d\omega.$$

There is another approach to solving the wave equation, which involves factoring the differential equation. We can rewrite the wave equation as

$$0 = \partial_t^2 u - \partial_x^2 u = (\partial_t^2 - \partial_x^2)u = (\partial_t - \partial_x)(\partial_t + \partial_x)u,$$

which implies that any solution  $u = u(x, t)$  of the wave equation satisfies either

$$0 = (\partial_t - \partial_x)u = \partial_t u - \partial_x u$$

or

$$0 = (\partial_t + \partial_x)u = \partial_t u + \partial_x u.$$

Now change variables to

$$(\bar{t}, \bar{x}) = (t - x, t + x).$$

One can check that is is a well-defined change of variables; in fact, it is just a rotation by  $\pi/4$  in the counter-clockwise direction and a rescaling by  $\sqrt{2}$ . This change of variables transforms the two differential equations into

$$0 = (\partial_t - \partial_x)u = \partial_{\bar{t}} u$$

and

$$0 = (\partial_t + \partial_x)u = \partial_{\bar{x}} u.$$

If we specialize to the case where  $g(x, 0) = 0$  (i.e. the initial velocity is zero) then the solution to (4) is given by

$$u(\bar{x}, \bar{t}) = \frac{1}{2}[u|_{\bar{t}=0} + u|_{\bar{x}=0}] = \frac{1}{2}[f(x+t) + f(x-t)].$$

In this case, we have recovered D'Alembert's principle: any solution to the wave equation is the sum of a left-moving wave and a right-moving wave.

## 6 The heat equation

This time the initial value problem we want to solve is

$$\begin{cases} \partial_t u(x, t) &= \partial_x^2 u(x, t) \\ u(x, 0) &= f(x). \end{cases} \quad (5)$$

This time  $u(x, t)$  is defined for all  $x$  and for  $t \geq 0$ . Again, we will need some decay conditions on  $f$ , which one can think of as boundary conditions.

Taking the Fourier transform in the  $x$ -variable, we obtain the equation

$$\hat{u}'(\omega, t) = -\omega^2 \hat{u}(\omega, t),$$

with the initial condition

$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

This ODE (with parameters) has the solution

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-\omega^2 t},$$

so taking the inverse Fourier transform we have

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} \hat{f}(\omega) e^{-\omega^2 t} d\omega.$$

This time, we can write our solution in a very nice form. The first key observation is that  $\hat{u}$  is the product of two Fourier transforms we already know. Indeed,  $\hat{u}$  is just  $\hat{f}$  multiplied by a rescaled Gaussian distribution. More precisely, using equation (2) we have

$$e^{-\omega^2 t} = e^{-\frac{(\omega\sqrt{2t})^2}{2}} = \sqrt{2\pi} \Phi(\omega\sqrt{2t}) = \sqrt{2\pi} \hat{\Phi}_{\sqrt{2t}}(\omega).$$

Plugging this into our formula for  $\hat{u}$ , we find

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-\omega^2 t} = \sqrt{2\pi} \hat{f}(\omega) \cdot \hat{\Phi}_{\sqrt{2t}}(\omega) = \mathcal{F}(f * \Phi_{\sqrt{2t}})(\omega).$$

Taking the inverse Fourier transform, we have

$$u(x, t) = f * \Phi_{\sqrt{2t}}(x, t) = \Phi_{\sqrt{2t}} * f(x, t) = \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy.$$

Thus we have written our solution to (5) as a convolution with a time-dependent Gaussian distribution. The function

$$?(x, y, t) := \frac{1}{\sqrt{2t}} \cdot e^{-\frac{(x-y)^2}{4t}}$$

is called the heat kernel, or the Gauss kernel. Notice that the ? decays exponentially as  $t \rightarrow \infty$ .

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