Solutions to the Midterm Exam, Math 3150
November 6, 2003

1. (12 points) Solve the following wave equation.

$$\begin{cases} \partial_t^2 u &= \partial_x^2 u + \partial_y^2 u \\ 0 &= u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) \\ u(x,y,0) &= 0 \\ \partial_t u(x,y,0) &= \sin(2x)\sin(4y) \end{cases}$$

The general solution to this equation is given by

$$u(x,y,t) = \sum_{n,m} \sin(nx)\sin(my)[a_{n,m}\cos(\sqrt{n^2+m^2}t) + b_{n,m}\sin(\sqrt{n^2+m^2}t)],$$

where

$$u(x,y,0) = \sum_{n,m} a_{n,m}\sin(ny)\sin(my), \quad \partial_t u(x,y,0) = \sum_{n,m} b_{n,m}\sqrt{n^2+m^2}\sin(nx)\sin(my).$$

Because $u(x,y,0) = 0$, we must have $a_{n,m} = 0$ for all n and m. Next we match the Fourier coefficients for $\partial_t u(x,y,0)$:

$$\sin(2x)\sin(4y) = \partial_t u(x,y,0) = \sum_{n,m} b_{n,m}\sqrt{n^2+m^2}\sin(nx)\sin(my).$$

This means $b_{n,m} = 0$ unless $n = 2$ and $m = 4$. In this last case, we have

$$1 = \sqrt{2^2+4^2}\,b_{2,4}, \text{ or } b_{2,4} = \frac{1}{\sqrt{20}}.$$

Finally, the solution u is

$$u(x,y,t) = \frac{1}{\sqrt{20}}\sin(2x)\sin(4y)\sin(\sqrt{20}t).$$

2. (12 points) Find the harmonic funtion u (i.e. $\Delta u = 0$) on the unit disc D with the boundary values

$$u(1,\theta) = f(\theta) = \cos(3\theta) - \sin(2\theta).$$

The general form of the harmonic function u is

$$u(r,\theta) = a_0 + \sum r^n[a_n\cos(n\theta) + b_n\sin(n\theta)]$$

where

$$f(\theta) = a_0 + \sum[a_n\cos(n\theta) + b_n\sin(n\theta)].$$

Our boundary data is already in Fourier series:

$$\cos(3\theta) - \sin(2\theta) = a_0 + \sum[a_n\cos(n\theta) + b_n\sin(n\theta)],$$

which implies that

$$a_3 = 1, \qquad b_2 = -1$$

and all the other coefficients are zero. Thus

$$u(r,\theta) = r^3\cos(3\theta) - r^2\sin(2\theta).$$

3. Consider the heat equation

$$\partial_t u = \partial_x^2 u, \quad u(0,t) = 0, \quad u(1,t) = 1, \quad u(x,0) = f(x) = x - 2\sin(3\pi x).$$

(a) (4 points) What is the steady state (i.e. the limit of $u(x,t)$ as $t \to \infty$) of this equation? The steady state solution u_0 is indepenent of time. Thus, $u_0 = u_0(x)$ satisfies

$$u_0'' = 0,$$

and thus it must be a linear function. We must also have

$$u_0(0) = 0, \qquad u_0(1) = 1,$$

and so

$$u_0(x) = x.$$

(b) (4 points) What are the modified initial and boundary conditions after you subtract off the steady state solution?
Let $\tilde{u} = u - u_0$. Then the boundary data is given by

$$\tilde{u}(0,t) = u(0,t) - u_0(0) = 0, \qquad \tilde{u}(1,t) = u(1,t) - u_0(1) = 0.$$

Similarly, the initial condition is given by

$$\tilde{u}(x,0) = u(x,0) - u_0(x) = -2\sin(3\pi x).$$

(c) (4 points) What is the solution to this heat equation with modified initial and boundary conditions?
The general solution to this heat equation is

$$\tilde{u}(x,t) = \sum a_n e^{-n^2\pi^2 t}\sin(n\pi x),$$

where

$$-2\sin(3\pi x) = \tilde{u}(x,0) = \sum a_n \sin(n\pi x).$$

Matching the coefficients, we see that $a_3 = -2$ and all the other coefficients are zero. Thus

$$\tilde{u}(x,t) = -2e^{-9\pi^2 t}\sin(3\pi x).$$

(d) (4 points) What is the solution to the original heat equation?
$u(x,t) = \tilde{u}(x.t) + u_0(x) = x - 2e^{-9\pi^2 t}\sin(3\pi x)$

4. Consider the heat equation

$$\partial_t u = \partial_x^2 u, \quad u(0,t) = 0 = \partial_x u(1,t), \quad u(x,0) = \sin((\pi x)/2).$$

(a) (3 points) Separate variables by writing $u(x,t) = X(x)T(t)$. What ODEs do X and T satisfy? (You may assume that the separation constant is negative.)
Let $u(x,t) = X(x)T(t)$. Then the differential equation becomes

$$XT' = X''T,$$

or

$$\frac{T'}{T}(t) = \frac{X''}{X}(x) = -\lambda^2,$$

for some constant λ.

(b) (3 points) What are the boundary conditions on X? Use these boundary conditions to find the form of X. (Hint: the right endpoint will tell you the possible values of the separation constant, but you have to be a little careful in finding these possible values.)
Because $u(0,t) = 0$, we must have $X(0) = 0$. Similarly, $\partial_x u(1,t) = 0$ forces $X'(1) = 0$. So the boundary conditions on X are

$$X(0) = 0, \qquad X'(1) = 0.$$

The general form of X solving $X'' = -\lambda^2 X$ is

$$X(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x).$$

Evaluating at $x = 0$ we see

$$0 = X(0) = c_1,$$

so we may as well take $X(x) = \sin(\lambda x)$. Evaluating at $x = 1$, we see

$$0 = X'(1) = \lambda\cos(\lambda),$$

which forces λ to be a *half* integer multiple of π. We can write this in general as

$$\lambda = \lambda_n = \frac{(2n+1)\pi}{2},$$

and so

$$X(x) = \sin(\frac{(2n+1)\pi x}{2}).$$

(c) (3 points) What is the general form of the solution to the heat equation with these boundary conditions? In general, we have

$$T(t) = a_n e^{-\lambda_n^2 t} = a_n e^{-(2n+1)^2\pi^2 t/4},$$

and so

$$u(x,t) = \sum_n a_n e^{-(2n+1)^2\pi^2 t/4} \sin(\frac{(2n+1)\pi x}{2}),$$

where

$$u(x,0) = \sum a_n \sin(\frac{(2n+1)\pi x}{2}).$$

(d) (1 point) What is the solution to this equation with the given initial data $u(x,0) = \sin((\pi x)/2)$? We match the Fourier series for the initial data:

$$\sin((\pi x)/2) = u(x,0) = \sum a_n \sin(\frac{(2n+1)\pi x}{2}),$$

so $a_0 = 1$ and the rest of the coefficients are zero. Putting this all together, we see

$$u(x,t) = e^{-\pi^2 t/4} \sin((\pi x)/2).$$