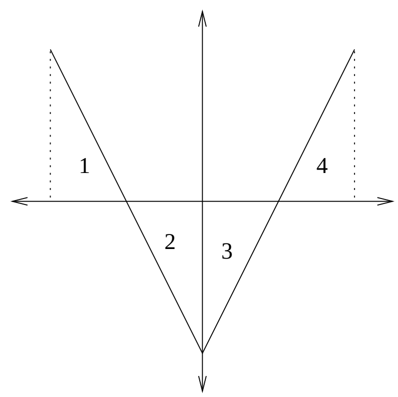
Midterm Exam Solutions, Math 3150
October 3, 2003

1. (8 points) Let $f(x) = 2|x| - 1$ for $-1 \leq x \leq 1$. Find the Fourier series for f. Think before you compute.

   Notice that f is an even function, so $b_k = 0$ for all k. Next we compute a_0:

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \int_0^1 (2x-1)dx = (x^2-1)\Big|_0^1 = 0.$$

   Another way to do this computation is to look at the graph of f (see the figure below).



   Notice that the graph consists of four triangles, labeled 1 through 4 in the figure. The integral $\int_{-1}^{1} f(x)dx$ is the area of triangles 1 and 4, minus the area of triangles 2 and 3. Since all these triangles are congruent and have equal area, the total integral is zero.

   Finally, we compute a_k:

$$\begin{aligned} a_k &= \int_{-1}^{1} f(x)\cos(k\pi x)dx = 2\int_0^1 (2x-1)\cos(k\pi x)dx \\ &= 4\int_0^1 x\cos(k\pi x)dx - 2\int_0^1 \cos(k\pi x)dx = 4\int_0^1 x\cos(k\pi x)dx \\ &= 4[\frac{x}{k\pi}\sin(k\pi x)\Big|_0^1 - \frac{1}{k\pi}\int_0^1 \sin(k\pi x)dx] = \frac{4}{k^2}\pi^2\cos(k\pi x)|_0^1 \\ &= \frac{4}{k^2\pi^2}((-1)^k - 1). \end{aligned}$$

   This last quantity is 0 if k is even and $-8/(k^2\pi^2)$ if k is odd. Thus

$$f(x) = \sum_{k \text{ odd}} \frac{-8}{k^2\pi^2}\cos(k\pi x) = \sum_1^\infty \frac{-8}{\pi^2(2k+1)^2}\cos((2k+1)\pi x).$$

2. (8 points) Let $f(x) = \sin x \cos x$ for $-\pi \leq x \leq \pi$. Find the Fourier series of f. Think before you compute.

   Recall the trig identity

$$\sin(2x) = 2\sin(x)\cos(x).$$

   Thus,

$$f(x) = \sin(x)\cos(x) = \frac{1}{2}\sin(2x),$$

   which is already in Fourier series form. Incidentally, you could have derived this identity from the identity

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

   which was on the front of the exam, by taking $y = x$.

   Also, you really did need to use a trig identity here. A function like $\cos^3 x \sin x$ is not in the form of a Fourier series.

3. Let f be a piecewise continuous function for $-\pi \le x \le \pi$ with a Fourier series

$$f(x) = a_0 + \sum_1^\infty [a_k \cos(kx) + b_k \sin(kx)].$$

(a) (3 points) What can you say about f if $b_k = 0$ for all k? Why?
If $b_k = 0$ then f has the form

$$f(x) = a_0 + \sum_1^\infty a_k \cos(kx),$$

which is a sum of even functions. Therefore, f is even.

(b) (3 points) What can you say about f if $a_0 = 0$? Why?
Recall that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx,$$

which is the average value of f. Thus $a_0 = 0$ if and only if the average value of f is zero.

(c) (3 points) If $g(x) = f(x) - f(-x)$, what can you say about the Fourier series of g? Why?
Notice that

$$g(-x) = f(-x) - f(x) = -g(x),$$

so g is odd. Because a_k is the integral of $g(x)\cos(kx)$, which is an odd function, we must have $a_k = 0$ for $k = 0, 1, 2, 3, \ldots$.
On this problem, a lot of people assumed that f was even, which was not my intent. However, I realized while grading that you could have read the question that way, so I let it slide if you said something like "f is an even function, so g must be zero."

4. For which of the following differential equations does the principle of superposition hold (i.e. if u and $\tilde{u}$ are solutions to the differential equation then so are $u + \tilde{u}$ and au for any constant a)? Be sure to justify your answers!

The general principle is that superposition holds for linear, homogeneous equations.

(a) (3 points) $u_{xt} = u_x - u_t$
Yes; this is a linear, homogeneous equation.

(b) (3 points) $u_{xx} = x + u_t$
No; this is linear equations, but it's not homogeneous. For instance, if $u_{xx} = x + u_t$ and $\tilde{u}_{xx} = x + \tilde{u}_t$, then

$$(u + \tilde{u})_{xx} = 2x + (u + \tilde{u})_t,$$

which is not the equation we want.

(c) (3 points) $u_{tt} = u_x u_t$
No; this is a nonlinear equation.

5. Consider the wave equation

$$(*) \quad \partial_t^2 u = \partial_x^2 u, \qquad u(0,t) = 0 = u(\pi,t), \qquad u(x,0) = 0, \qquad \partial_t u(x,0) = \sin(2x)$$

(a) (4 points) Begin by looking for a solution of the form $u(x,t) = v(x)w(t)$. What differential equations must v and w satisfy?
Notice that $\partial_x u(x,t) = \partial_x(v(x)w(t)) = v'(x)w(t)$. Similarly, we have

$$\partial_t^2 u = vw'' \qquad \partial_x^2 u = v''w.$$

Thus, by the equation we have

$$v''(x)w(t) = v(x)w''(t),$$

which we can rearrange to read

$$\frac{v''}{v}(x) = \frac{w''}{w}(t).$$

The key point is that the right hand side of this equation depends on t alone, which the left hand side depends on x alone. Therefore, both sides of the equation must be equal to some constant λ. We then have

$$v'' = \lambda v \qquad w'' = \lambda w.$$

(b) (4 points) What are the boundary/initial conditions for v?
We have boundary conditions for v. Note that

$$0 = u(0,t) = v(0)w(t).$$

Unless $w \equiv 0$ (which means the string doesn't move), we must have

$$v(0) = 0.$$

Similarly, we have

$$v(\pi) = 0.$$

Incidentally, one can conclude from this information that $v(x) = a_n \sin(nx)$ for some positive integer n.

(c) (4 points) What are the boundary/initial conditions for w?
We have initial conditions for w. Note that

$$0 = u(x,0) = v(x)w(0),$$

from which we can conclude $w(0) = 0$. Similarly, we have

$$\sin(2x) = \partial_t u(x,0) = v(x)w'(0).$$

Therefore,

$$w'(0) = \frac{\sin(2x)}{v(x)}.$$

Because this is a constant, we may as well take

$$v(x) = \sin(2x).$$

(We are absorbing a constant into w by doing this, otherwise we'd have $v(x) = a\sin(2x)$.) This forces $\lambda = -4$, and so

$$w(t) = acos(2t) + b\sin(2t).$$

(d) (4 points) Write down the solution to $(*)$.
We already know that

$$u(x,t) = \sin(2x)(a\cos(2t) + b\sin(2t))$$

for some choice of a and b. By the initial conditions on w, we must have $a = 0$ and $b = 1/2$ (try it!), and so

$$u(x,t) = \frac{1}{2}\sin(2x)\sin(2t).$$