Practice Problems

Math 252

April 21, 2006

These problems are in no particular order.

1. Evaluate the following contour integrals.
   (a) $\int_{|z|=2} \frac{\sin z}{z-\pi/2} dz$
   (b) $\int_{|z|=2} \frac{\cos z}{(z-\pi/2)^4} dz$
   (c) $\int_{|z+1|=1/2} \frac{z}{1-z^2} dz$
   (d) $\int_{|z-1|=1/2} \frac{\sin(\pi z/2)}{e^z-1} dz$
   (e) $\int_{|z|=1} \frac{e^z}{1-\cos z} dz$
2. Suppose $f(z)$ is analytic on the region $\{|z| > 1\}$, and $|f(z)| \leq 5/|z|^2$.
   (a) Show that f has a pole at ∞, of order at most 2.
   (b) What can you say about the residue of f at ∞?
3. Let f and g be meromorphic functions on a region $D \subset \mathbb{C}$. Also, let $z_0 \in D$, let f have a zero of order 2 at z_0, and let g have a pole at z_0.
   (a) If the function $h(z) = f(z) \cdot g(z)$ is analytic at z_0, what can you say about the order of the pole of g at z_0?
   (b) If h is not only analytic, but $h(z_0) = 0$, what can you say about the pole of g at z_0?
   (c) In general, let k be the order of the pole of g at z_0. Compute the order of the pole/zero of h at z_0, depending on k. Be sure to distinguish between the cases when h has a pole and when h has a zero.
4. Compute the residues of the following functions at the given points.
   (a) $f(z) = \frac{e^z}{(z^2-4)^2}$, $z_0 = 2$
   (b) $f(z) = \tan z$, $z_0 = \pi/2$
   (c) $f(z) = \frac{1}{\cos z}$, $z_0 = \pi$
   (d) $f(z) = e^{1/z}$, $z_0 = 0$
5. Suppose f is analytic on the entire complex plane and the real part of f is positive. Show that f must be constant. (Hint: the exponential map.)
6. Suppose that f is analytic on the entire complex plane and that $|f(z)| \leq 5|z|^3$ for $|z| \geq 100$.
   (a) Show that f is a polynomial of degree at most 3.
   (b) Given that f is a cubic polynomial, it has the form $f(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$. What can you say about a_3? In particular, can you find a bound for it?
7. Find the maximum of $|\cos z|$ of the domain $\{z = x + iy \mid -\pi \leq x \leq \pi, 0 \leq y \leq 2\}$.
8. Find the Laurent series expansions for the following functions based at the given points. Also, give the radius of convergence of each series. (Note: a Taylor series is a special case of a Laurent series.)
   (a) $f(z) = \frac{1}{1-z^2}$, $z_0 = 2$
   (b) $f(z) = e^z - 1$, $z_0 = -1$
   (c) $f(z) = \cos z$, $z_0 = -\pi/2$
   (d) $f(z) = \frac{z}{z^2-4}$, $z_0 = 0$
9. Consider the function
$$f(z) = \frac{1}{z^3 - 9z}.$$
   (a) Find a Laurent series for f in the annulus $\{0 < |z| < 3\}$.

(b) Find a Laurent series for f in the annulus $\{|z| > 3\}$.

10. For each given function f and domain D, describe the image of the domain D under f.

    (a) $f(z) = z^3$, $D = \{1 \leq |z| \leq 8, \Re(z) > 0, \Im(z) > 0\}$

    (b) $f(z) = \cos z$, $D = \{x + iy \mid 0 \leq x \leq \pi/2, y < 0\}$

    (c) $f(z) = \log(z)$ (principal branch), $D = \{z \mid 1 \leq |z| \leq 2, \pi/4 \leq \arg(z) \leq 3\pi/4\}$

11. Compute the following definite integrals.

    (a) $\int_{-\infty}^{\infty} \frac{dx}{1+x+x^2}$

    (b) $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

    (c) $\int_{-\infty}^{\infty} \frac{xdx}{x^3-x^2+x-1}$ (Hint: $x - 1$ is a factor of the denominator.)

12. For each of the following functions f, describe the largest domain D on which f is analytic.

    (a) $f = \cot z$

    (b) $f = \frac{\sqrt{4-z^2}}{e^z-1}$

    (c) $f = \frac{1}{1+e^z}$

    (d) $f = |z|^2$

13. Let $u(x, y) = e^x \cos y$.

    (a) Verify that u is harmonic.

    (b) Find the minimum of u on the domain $-1 \leq x \leq 1$, $-\pi \leq y \leq 0$.

    (c) Find a harmonic conjugate v of u. (Hint: you might recognize u as the real part of a familiar analytic function.)

14. Evaluate $\int_\gamma f(z)dz$ for each of the following contours γ, when f is given by

$$f(x) = \frac{z}{z^2 - 1}.$$

    (a) $\gamma = \{|z| = 1/2\}$

    (b) $\gamma = \{|z - 1| = 1\}$

    (c) $\gamma = \{|z + 1| = 1\}$

    (d) $\gamma = \{|z| = 2\}$

15. Let $f(z)$ be a meromorphic function in the disc $D = \{|z| < 2\}$, with singularities at $z_1 = 0, z_2 = 1/2, z_3 = -1/2$. Suppose the residue at 0 is 1, the residue at $1/2$ is -3, and the residue at $-1/2$ is 2. Show that there is a function $g(z)$, which is analytic on $D \backslash [-1/2, 1/2]$, such that $g' = f$. (Hint: the fundamental theorem of calculus.)