Solutions to the Practice Problems

Math 252

April 26, 2006

1. Evaluate the following contour integrals.

   (a) $\int_{|z|=2} \frac{\sin z}{z-\pi/2} dz$

   We use the Cauchy integral formula:

   $$\int_{|z|=2} \frac{\sin z}{z-\pi/2} dz = 2\pi i \sin(\pi/2) = 2\pi i.$$

   (b) $\int_{|z|=2} \frac{\cos z}{(z-\pi/2)^4} dz$

   This time we use the Cauchy integral formula for derivatives:

   $$\int_{|z|=2} \frac{\cos z}{(z-\pi/2)^4} dz = \frac{2\pi i}{3!} (\cos z)'''|_{z=\pi/2} = \frac{\pi i}{3} \sin(\pi/2) = \frac{\pi i}{3}.$$

   (c) $\int_{|z+1|=1/2} \frac{z}{1-z^2} dz$

   We will use the residue theorem. The integrand $z/(1-z^2)$ has simple poles at $z = \pm 1$, so

   $$\int_{|z+1|=1/2} \frac{z}{1-z^2} dz = 2\pi i \text{Res}(z/(1-z^2), -1) = 2\pi i \lim_{z\to -1} \frac{z(z+1)}{1-z^2} = 2\pi i \lim_{z\to -1} \frac{z}{1-z} = \pi i.$$

   (d) $\int_{|z-1|=1/2} \frac{\sin(\pi z/2)}{e^z - 1} dz$

   Notice that the disc $|z-1| \leq 1/2$ does not contain the origin $z = 0$, so the integrand $\sin(\pi z/2)/(e^z - 1)$ is analytic inside this disc. Thus, by Cauchy's theorem, the integral is 0.

   (e) $\int_{|z|=1} \frac{e^z}{1-\cos z} dz$

   We will again use the residue theorem:

   $$\int_{|z|=1} \frac{e^z}{1-\cos z} dz = 2\pi i \text{Res}(\frac{e^z}{1-\cos z}, 0).$$

   To compute the residue, first observe that the integrand has a double pole at the origin, so we can write

   $$\frac{e^z}{1-\cos z} = \frac{b_2}{z^2} + \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n.$$

   Now cross-multiply and use the Taylor series expansions for e^z and $1 - \cos z$:

   $$1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots = (\frac{z^2}{2} - \frac{z^4}{4!} + \frac{z^6}{6!} - + \cdots)(\frac{b_2}{z^2} + \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \cdots) = \frac{b_2}{2} + \frac{b_1}{2} z + \cdots.$$

   Matching the z terms, we see $\text{Res}(e^z/(1-\cos z), 0) = b_1 = 2$, so

   $$\int_{|z|=1} \frac{e^z}{1-\cos z} dz = 4\pi i.$$

2. Suppose $f(z)$ is analytic on the region $\{|z| > 1\}$, and $|f(z)| \leq 5/|z|^2$.

   (a) Show that f has a pole at ∞, of order at most 2.

   We change variables by letting $w = 1/z$

   (b) What can you say about the residue of f at ∞?

3. Let f and g be meromorphic functions on a region $D \subset \mathbb{C}$. Also, let $z_0 \in D$, let f have a zero of order 2 at z_0, and let g have a pole at z_0.

   (a) If the function $h(z) = f(z) \cdot g(z)$ is analytic at z_0, what can you say about the order of the pole of g at z_0?

Write the Laurent series expansion of f and g as

$$f = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \qquad g(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n + \sum_{n=1}^{\infty} B_n (z - z_0)^{-n}.$$

Because f has a zero of order 2 at z_0, we know $b_n = 0$ for all n, and $a_0 = 0, a_1 = 0$. Multiplying the two series, we get

$$\begin{aligned} f \cdot g(z) &= (a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots)(\frac{B_1}{z - z_0} + \frac{B_2}{(z - z_0)^2} + \frac{B_3}{(z - z_0)^3} + \dots) + (a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots \\ &= a_2(\frac{B_3}{z - z_0} + \frac{B_4}{(z - z_0)^2} + \dots) + a_3(\frac{B_4}{z - z_0} + \frac{B_5}{(z - z_0)^2} + \dots) + \dots \end{aligned}$$

The rest of the terms in the sum are positive powers of $z - z_0$, so that part of the Laurent series is analytic. The point is that in order for the product $f \cdot g$ to be analytic, we need $B_3 = 0$, $B_4 = 0$, $B_5 = 0$, and so on. In other words, the order of the pole of g is at most 2.

(b) If h is not only analytic, but $h(z_0) = 0$, what can you say about the pole of g at z_0?
So far, we've concluded that $g = B_2/(z - z_0)^2 + B_1/(z - z_0) + \sum_{n=0}^{\infty} A_n (z - z_0)^n$, and so

$$h(z) = f(z)g(z) = a_2 B_2 + (a_3 B_2 + a_2 B_1)(z - z_0) + \dots$$

If $h(z_0) = 0$, this forces $a_2 B_2 = 0$. However, we already know $a_2 \neq 0$, which forces $B_2 = 0$. In other words, z_0 must be a simple pole of g.

(c) In general, let k be the order of the pole of g at z_0. Compute the order of the pole/zero of h at z_0, depending on k. Be sure to distinguish between the cases when h has a pole and when h has a zero.
Let g have a pole of order k. Then it has a Laurent expansion

$$g = \frac{B_k}{(z - z_0)^k} + \dots \frac{B_1}{z - z_0} + A_0 + A_1(z - z_0) + \dots.$$

We have already treated the cases $k = 2$ (when h has neither a zero nor a pole) and $k = 1$ (when h has a simple zero). If $k > 2$ then h has a Laurent expansion

$$h(z) = f(z)g(z) = \frac{a_2 B_K}{(z - z_0)^{k-2}} + \dots,$$

and so h has a pole of order $k - 2$. This covers all the possibilities for k.

4. Compute the residues of the following functions at the given points.

(a) $f(z) = \frac{e^z}{(z^2-4)^2}$, $z_0 = 2$
The pole of f at $z_0 = 2$ is a double pole, so the Laurent expansion of f is

$$f(z) = \frac{b_2}{(z - 2)^2} + \frac{b_1}{z - 2} + a_0 + a_1(z - 2) + a_2(z - 2)^2 + \dots = \frac{e^z}{(z^2 - 4)^2} = \frac{e^z}{(z - 2)^2(z + 2)^2}.$$

We first write out the Taylor series for e^z centered at 2:

$$e^z = \sum_{n=0}^{\infty} \frac{(e^z)^{(n)}\big|_{z=2}}{n!}(z - 2)^n = e^2 \sum_{n=0}^{\infty} \frac{(z - 2)^n}{n!}.$$

Next we rewrite $(z + 2)^2$:

$$(z + 2)^2 = ((z - 2) + 4)^2 = (z - 2)^2 + 8(z - 2) + 16;$$

this is the Taylor series expansion of $(z + 2)^2$ centered at $z_0 = 2$. Finally, we plug in the expansions we just found into the Laurent series for f and cross-multiply:

$$e^z = e^2 \sum_{n=0}^{\infty} \frac{(z - 2)^n}{n!} = f(z)(z-2)^2(z+2)^2 = (z-2)^2((z-2)^2+8(z-2)+16)(\frac{b_2}{(z - 2)^2} + \frac{b_1}{z - 2} + a_0 + a_1(z-2) + a_2(z-2)^2 + \dots$$

Grouping terms, we see

$$e^2 \sum_{n=0}^{\infty} \frac{(z - 2)^2}{n!} = 16b_2 + (8b_2 + 16b_1)(z - 2) + \dots,$$

which tells us $16b_2 = e^2$ and $8b_2 + 16b_1 = e^2$. Solving for b_1, we get $\mathrm{Res}(f(z), 2) = b_1 = e^2/32$.

(b) $f(z) = \tan z$, $z_0 = \pi/2$

Notice $\tan z = \sin z / \cos z$, so $\tan z$ has a simple pole at $z = \pi/2$. Thus

$$\text{Res}(\tan z, \pi/2) = \lim_{z\to\pi/2}(z-\pi/2)\tan z = \lim_{z\to\pi/2}\frac{(z-\pi/2)(1-(z-\pi/2)^2/2+\ldots)}{-(z-\pi/2)+(z-\pi/2)^3/3!+\ldots} = -1.$$

(c) $f(z) = \frac{1}{\cos z}$, $z_0 = \pi$

Notice that $\cos(0) = -1 \neq 0$, so f is analytic near π. Thus the residue is 0.

(d) $f(z) = e^{1/z}$, $z_0 = 0$

We can write out the Laruent series for $e^{1/z}$ centered at 0:

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{(1/z)^n}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots$$

Thus the residue is $\text{Res}(e^{1/z}, 0) = 1$.

5. Suppose f is analytic on the entire complex plane and the real part of f is positive. Show that f must be constant. (Hint: the exponential map.)

Let $g(z) = e^{-f(z)}$ and notice that g is analytic on the entire complex plane (it is the composition of analytic functions). We use $\Re(f(z)) > 0$ to estimate $|g(z)|$:

$$|g(z)| = |e^{-f(z)}| = e^{-\Re(f(z))} \leq e^0 = 1.$$

Thus g is a bounded function which is analytic on the entire complex plane, and so it is constant by Liouville's theorem. This implies that f is constant.

6. Suppose that f is analytic on the entire complex plane and that $|f(z)| \leq 5|z|^3$ for $|z| \geq 100$.

(a) Show that f is a polynomial of degree at most 3.

We will show the fourth derivative of f is identically zero using the Cauchy integral formula. Fix a point z_0 in the complex plane. Then, for $R > \max(|z_0|, 100)$, we have

$$\begin{aligned}
|f^{(4)}(z_0)| &= \frac{4!}{2\pi}\left|\int_{|z|=R}\frac{f(z)}{(z-z_0)^5}dz\right| \leq \frac{4!}{2\pi}\int_{|z|=R}\frac{|f(z)||dz|}{|z-z_0|^5} \\
&\leq \frac{12}{\pi}\int_{|z|=R}\frac{5|z|^3|dz|}{(|z|-|z_0|)^5} = \frac{12}{\pi}\cdot\frac{5R^3}{(R-|z_0|)^5}\int_0^{2\pi}Rd\theta \\
&= \frac{120R^4}{(R-|z_0|)^5}.
\end{aligned}$$

This last quantity goes to zero when $R \to \infty$, so $f^{(4)}(z_0) = 0$. However, z_0 was arbitrary, so the fourth derivative is identically zero, which means that f is a cubic polynomial.

(b) Given that f is a cubic polynomial, it has the form $f(z) = a_3z^3 + a_2z^2 + a_1z + a_0$. What can you say about a_3? In particular, can you find a bound for it?

We will write out f'''. First, we have

$$f''' = (a_3z^3 + z_2z^2 + a_1z + a_0)''' = 6a_3.$$

Next we estimate the third derivative using the Cauchy integral formula:

$$\begin{aligned}
6|a_3| &= |f'''(0)| = \frac{3!}{2\pi}\left|\int_{|z|=R}\frac{f(z)}{z^4}dz\right| \leq \frac{3}{\pi}\int_{|z|=R}\frac{|f(z)||dz|}{|z|^4} \\
&\leq \frac{3}{\pi}\int_{|z|=R}\frac{5R^3|dz|}{R^4} = \frac{3}{\pi}\cdot\frac{5R^3}{R^4}\cdot 2\pi R = 30.
\end{aligned}$$

Thus we have $6|a_3| \leq 30$, or $|a_3| \leq 5$.

7. Find the maximum of $|\cos z|$ of the domain $\{z = x + iy \mid -\pi \leq x \leq \pi, 0 \leq y \leq 2\}$.

We first compute $|\cos z|$:

$$|\cos z|^2 = \frac{1}{4}|e^{ix-y}+e^{-ix+y}|^2 = \frac{1}{4}|e^y(\cos x + i\sin x) + e^{-y}(\cos x - i\sin x)|^2 = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y = \cos^2 x + \sinh^2 y.$$

Here we note that $\cos x$ is a periodic function, oscillating between -1 and 1, while $\sinh y$ is monotone increasing, with $\sinh(0) = 0$. By the maximum principle, the maximum of $|\cos z|$ occur on the boundary, so we only need to check $|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$ on the edges $x = \pm\pi$, $y = 0$, and $y = 2$. On the edge $y = 0$, we have $|\cos z| = |\cos x|$, which has a maximal value of 1 at $x = 0, \pi, -\pi$. On the edge $y = 2$, we have $|\cos z| = \sqrt{\cos^2 x + \sinh^2(2)}$, which has a maximum of $\sqrt{1 + \sinh^2(2)}$ at $x = 0, \pi, -\pi$. On the edges $x = \pm\pi$, we have $|\cos z| = \sqrt{1 + \sinh^2 y}$, which has a maximum of $\sqrt{1 + \sinh^2(2)}$ at $y = 2$. So the maximum of $|\cos z|$ is $\sqrt{1 + \sinh^2(2)}$, which occurs along the $y = 2$ edge at $x = 0, \pi, -\pi$.

8. Find the Laurent series expansions for the following functions based at the given points. Also, give the radius of convergence of each series. (Note: a Taylor series is a special case of a Laurent series.)

(a) $f(z) = \frac{1}{1-z^2}$, $z_0 = 2$
We first rewrite f:

$$f(z) = \frac{1}{1-z^2} = \frac{1/2}{1+z} + \frac{1/2}{1-z}.$$

Next we need to compute derivatives of $1/(1-z)$ and $1/(1+z)$:

$$\left[\frac{1}{1+z}\right]^{(n)} = \frac{(-1)^n n!}{(z+1)^{n+1}}, \left[\frac{1}{1-z}\right]^{(n)} = \frac{n!}{(1-z)^{n+1}}.$$

Evaluating derivatives, we have

$$f(z) = \frac{1/2}{1+z} + \frac{1/2}{1-z} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^n}{3^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n(1-3^{-n-1}).$$

(b) $f(z) = e^z - 1$, $z_0 = -1$
Note that $f^{(n)}(-1) = e^z|_{z=-1} = 1/e$, so

$$f(z) = \frac{1}{e} - 1 + \sum_{n=1}^{\infty} \frac{(z+1)^n}{n!e}.$$

(c) $f(z) = \cos z$, $z_0 = -\pi/2$

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos^{(n)}(\pi/2)(z-\pi/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(z-\pi/2)^{2n+1}}{(2n+1)!}$$

(d) $f(z) = \frac{z}{z^2-4}$, $z_0 = 0$
We rewrite f using partial fractions:

$$f(z) = \frac{z}{z^2-4} = z\left[\frac{1/4}{z-2} - \frac{1/4}{z+2}\right].$$

Each of these is a geometric series:

$$\frac{1}{z-2} = -\frac{1}{2(1-z/2)} = -\frac{1}{2}\sum_{n=0}^{\infty}(z/2)^n$$

and

$$\frac{1}{z+2} = \frac{1}{2(1-(-z/2))} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(z/2)^n.$$

Putting all this together, we see

$$f(z) = \frac{1}{z}\left[-\frac{1}{8}\sum_{n=0}^{\infty}(z/2)^n - \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n(z/2)^n\right] = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{z^{2n}}{2^{2n+1}}.$$

9. Consider the function

$$f(z) = \frac{1}{z^3 - 9z}.$$

(a) Find a Laurent series for f in the annulus $\{0 < |z| < 3\}$.
First we write out a partial fractions decomposition of f:

$$f(z) = \frac{1}{z^3 - 9z} = \frac{1}{z(z-3)(z+3)} = \frac{1}{z}\left[\frac{1/6}{z-3} - \frac{1/6}{z+3}\right].$$

We can write each of these terms as a geometric series:

$$\frac{1/6}{z-3} = -\frac{1}{6}\cdot\frac{1}{3-z} = -\frac{1}{18}\cdot\frac{1}{1-(z/3)} = -\frac{1}{18}\sum_{n=0}^{\infty}(z/3)^n$$

and

$$\frac{1/6}{z+3} = \frac{1}{6}\cdot\frac{1}{3+z} = \frac{1}{18}\cdot\frac{1}{1-(-z/3)} = \frac{1}{18}\sum_{n=0}^{\infty}(-1)^n(z/3)^n.$$

Multiplying everything together, we have have

$$f(z) = \frac{1}{z}\left[-\frac{1}{18}\sum_{n=0}^{\infty}(z/3)^n - \frac{1}{18}\sum_{n=0}^{\infty}(-1)^n(z/3)^n\right] = -\frac{1}{9}\sum_{n=0}^{\infty}(z/3)^{2n+1}.$$

(b) Find a Laurent series for f in the annulus $\{|z| > 3\}$.
We will use the same partial fractions decomposition as before, but we have to rewrite our geometric series:

$$\frac{1/6}{z-3} = \frac{1}{6z}\cdot\frac{1}{1-3/z} = \frac{1}{6z}\sum_{n=0}^{\infty}(3/z)^n$$

and

$$\frac{1/6}{z+3} = \frac{1}{6z}\cdot\frac{1}{1-(-3/z)} = \frac{1}{6z}\sum_{n=0}^{\infty}(-1)^n(3/z)^n.$$

Again, we multiply everything together:

$$f(z) = \frac{1}{z}\left[\frac{1/6}{z-3} - \frac{1/6}{z+3}\right] = \frac{1}{36z^3}\left[\sum_{n=0}^{\infty}(3/z)^n - \sum_{n=0}^{\infty}(-1)^n(3/z)^n\right] = \frac{1}{18}\sum_{n=0}^{\infty}(3/z)^{2n+4}.$$

10. For each given function f and domain D, describe the image of the domain D under f.

(a) $f(z) = z^3$, $D = \{1 \le |z| \le 8, \Re(z) > 0, \Im(z) > 0\}$
Recall that $f(z) = z^3$ triples angles and cubes lengths. We can descibe D as the domain with $1 \le |z| \le 8$ and $0 < \arg(z) < \pi/2$. If we denote $w = f(D)$, we can describe $f(D)$ as the domain with $1 \le |w| \le 8^3 = 512$ and $0 < \arg(w) < 3\pi/2$.

(b) $f(z) = \cos z$, $D = \{x + iy \mid 0 \le x \le \pi/2, y < 0\}$
We rewrite $\cos z$: if $z = x + iy$ then

$$\begin{aligned}\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{ix-y} + e^{-ix+y}) = \frac{1}{2}(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)) \\ &= \frac{1}{2}(\cos x(e^{-y} + e^{y}) + i\sin x(e^{-y} - e^{y})) = \cos x\cosh y - i\sin x\sinh y.\end{aligned}$$

Now we plug in our constraints on x and y. Because $y < 0$ we have $\cosh y > 1$ and $\sinh y < 0$. Also, $0 \le x \le \pi/2$, so $\cos x$ and $\sin x$ take all values between 0 and 1. Putting this together, we see that $\Re(\cos z) = \cos x \cosh y$ can be any positive number, and $\Im(\cos z) = -\sin x \sinh y$ can also be any positive number. Thus we se that $f(D)$ is the first quadrant.

(c) $f(z) = \log(z)$ (principal branch), $D = \{z \mid 1 \le |z| \le 2, \pi/4 \le \arg(z) \le 3\pi/4\}$
Recall that the principal branch of the logarithm satisfies

$$\log z = \ln|z| + i\arg z,$$

so on D we have $0 \le \Re(\log(z)) \le \ln 2$ and $\pi/4 \le \Im(\log(z)) \le 3\pi/4$. This domain is a rectangle.

11. Compute the following definite integrals.

(a) $\int_{-\infty}^{\infty} \frac{dx}{1+x+x^2}$
The function

$$f(z) = \frac{1}{1+z+z^2} = \frac{1}{(z+1/2+i\sqrt{3}/2)(z+1/2-i\sqrt{3}/2)}$$

is analytic except at $z = -1/2 \pm i\sqrt{3}/2$, where it has simple poles. Also, $|f(z)| \leq 1/(2|z|^2)$ for large $|z|$, so we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x+x^2} = 2\pi i \mathrm{Res}(f, -1/2+i\sqrt{3}/2) = 2\pi i \lim_{z\to -1/2+i\sqrt{3}/2} \frac{1}{z+1/2+i\sqrt{3}/2} = \frac{4\pi}{\sqrt{3}}.$$

(b) $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$
If $f(z) = (1+z^4)^{-1}$, then f is analytic except at

$$z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4},$$

where it has simple poles. Also, $|f(z)| \leq 1/(2|z|^4)$ for large $|z|$. So we can write the definite integral in terms of residues at $e^{i\pi/4}$ and $e^{3\pi i/4}$. We will compute these residues, using the fact that the numerator of f doesn't vanish while the denominator has simple zeroes

$$\mathrm{Res}(f, e^{i\pi/4}) = \frac{1}{(1+z^4)'|_{z=e^{i\pi/4}}} = \frac{1}{4(e^{3\pi i/4})} = \frac{1}{4}e^{-3\pi i/4}$$

and

$$\mathrm{Res}(f, e^{3i\pi/4}) = \frac{1}{(1+z^4)'|_{z=e^{3i\pi/4}}} = \frac{1}{4(e^{9\pi i/4})} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}e^{-\pi i/4}.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} &= 2\pi i[\mathrm{Res}(f, e^{i\pi/4}) + \mathrm{Res}(f, e^{3i\pi/4})] = 2\pi i \cdot \frac{1}{4}(e^{-3i\pi/4} + e^{-\pi i/4}) \\ &= \frac{\pi i}{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = \frac{\pi}{2}\cdot\frac{2}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

(c) $\int_{-\infty}^{\infty} \frac{x\,dx}{x^3-x^2+x-1}$ (Hint: $x-1$ is a factor of the denominator.)
We factor the denominator:

$$f(z) = \frac{1}{z^3-z^2+z-1} = \frac{1}{(z-1)(z^2+1)} = \frac{1}{(z-1)(z+i)(z-i)}.$$

This function is analytic everywhere except at $z = 1, i, -i$, where it has simple poles. We next compute the residues of f at the poles $z=1$ and $z=i$ (we won't need the residue at $z=-i$):

$$\mathrm{Res}(f, 1) = \lim_{z\to 1} \frac{z-1}{z^3-z^2+z-1} = \lim_{z\to 1}\frac{1}{z^2+1} = \frac{1}{2},$$

and

$$\mathrm{Res}(f, i) = \lim_{z\to i} \frac{z-i}{z^3-z^2+z-1} = \lim_{z\to i}\frac{1}{(z-1)(z+i)} = \frac{1}{(i-1)(i+i)} = \frac{-1+i}{4}.$$

We also notice that $|f(z)| \leq 1/(2|z|^3)$ for large $|z|$, so we can compute the integral in terms of the residues:

$$\int_{-\infty}^{\infty} \frac{dx}{x^3-x^2+x-1} = 2\pi \mathrm{Res}(f, i) + \pi i \mathrm{Res}(f, 1) = 2\pi i \cdot \frac{i-1}{4} + \frac{\pi i}{2} = -\frac{\pi i}{2} - \frac{\pi}{2} + \frac{\pi i}{2} = -\frac{\pi}{2}.$$

12. For each of the following functions f, describe the largest domain D on which f is analytic.

(a) $f = \cot z$
We have $\cot z = \frac{\cos z}{\sin z}$, which is analytic so long as $\sin z \neq 0$. The zeroes of $\sin z$ are $z = \pi\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots = \frac{(2n+1)\pi}{2}$, where n can be any integer. Thus $f = \cot z$ is analytic on $\mathbb{C}\backslash\{(2n+1)\pi/2\}$.

(b) $f = \frac{\sqrt{4-z^2}}{e^z-1}$
First notice that we can't have we can't have $e^z = 1$, which happens when $z = 2n\pi i$, for any integer n. Next we have to choose some branch cuts for the square root function. We have

$$\sqrt{4-z^2} = \sqrt{2-z}\sqrt{2+z} = e^{(1/2)\log(2-z)} \cdot e^{(1/2)\log(2+z)}.$$

We choose the branch cut for $\sqrt{2-z}$ to be the ray where $2-z \leq 0 \Leftrightarrow z \geq 2$, and we choose the branch cut for $\sqrt{2+z}$ to be the ray $2+z \leq 0 \Leftrightarrow z \leq -2$. Thus the function f is analytic on the domain

$$D = \mathbb{C}\backslash\{2n\pi i\} \cup \{z = x + i\cdot 0, x \leq -2\} \cup \{z = x + i \cdot 0, x \geq 2\}.$$

(c) $f = \frac{1}{1+e^z}$

This function is analytic so long as $e^z \neq -1$, which happens when $z = \pm\pi i, \pm 3\pi i, \pm 5\pi i, \ldots$. So the domain D on which f is analytic is

$$D = \mathbb{C} \backslash \{(2n+1)\pi i\}.$$

(d) $f = |z|^2$

This function does not satisfy the Cauchy Riemann equations anywhere, so it is not analytic on any domain.

13. Let $u(x, y) = e^x \cos y$.

(a) Verify that u is harmonic.

We take some derivatives:

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0,$$

and so u is harmonic.

(b) Find the minimum of u on the domain $-1 \leq x \leq 1$, $-\pi \leq y \leq 0$.

By the maximum principle (applied to $-u$), the minimum of u occurs on the boundary, so we only need to check the edges of the rectangle. On the edge $x = -1$, we have $u = e \cos y$, which has a minimum of $-e$ at $y = -\pi$. On the edge $x = -1$, we have $u = (\cos y)/e$, which has a minimum of $-1/e$ at $y = -\pi$. On the edge $y = 0$, we have $u = e^x$, which has a minimum of $1/e$ at $x = -1$. On the edge $y = -\pi$, we have $u = -e^x$, which has a minimum of $-e$ at $x = 1$. Thus the minimum value of u is $-e$, which occurs at $x = 1, y = -\pi$.

(c) Find a harmonic conjugate v of u. (Hint: you might recognize u as the real part of a familiar analytic function.)

Observe that $u = \Re(e^z)$, so we can choose the harmonic conjugate to be

$$v = \Im(e^z) = \Im(e^x(\cos y + i \sin y)) = e^x \sin y.$$

14. Evaluate $\int_\gamma f(z)dz$ for each of the following contours γ, when f is given by

$$f(x) = \frac{z}{z^2 - 1}.$$

(a) $\gamma = \{|z| = 1/2\}$

Notice that f is analytic except at $z = 1, z = -1$, neither of which is contained inside the contour $|z| = 1/2$. Thus, by Cauchy's theorem, $\int_{|z|=1/2} f(z)dz = 0$.

(b) $\gamma = \{|z - 1| = 1\}$

This time we use the residue theorem:

$$\int_{|z-1|=1} f(z)dz = 2\pi i \text{Res}(f, 1) = 2\pi i \lim_{z \to 1} (z - 1)\frac{z}{z^2 - 1} = 2\pi i \lim_{z \to 1} \frac{z}{z + 1} = \pi i.$$

(c) $\gamma = \{|z + 1| = 1\}$

Again, we use the residue theorem:

$$\int_{|z+1|=1} f(z)dz = 2\pi i \text{Res}(f, -1) = 2\pi i \lim_{z \to -1} (z + 1)\frac{z}{z^2 - 1} = 2\pi i \lim_{z \to -1} \frac{z}{z - 1} = \pi i.$$

(d) $\gamma = \{|z| = 2\}$

This time the contour includes both the singularities of f, so we need to sum the residues:

$$\int_{|z|=2} f(z)dz = 2\pi i(\text{Res}(f, 1) + \text{Res}(f, -1)) = 2\pi i(1/2 + 1/2) = 2\pi i.$$

15. Let $f(z)$ be a meromorphic function in the disc $D = \{|z| < 2\}$, with singularities at $z_1 = 0, z_2 = 1/2, z_3 = -1/2$. Suppose the residue at 0 is 1, the residue at $1/2$ is -3, and the residue at $-1/2$ is 2. Show that there is a function $g(z)$, which is analytic on $D \backslash [-1/2, 1/2]$, such that $g' = f$. (Hint: the fundamental theorem of calculus.)

We pick a basepoint $z_0 \in D \backslash [-1/2, 1/2]$, and define

$$g(z) := \int_\gamma f(z) dz,$$

where γ is any curve in $D \backslash [-1/2, 1/2]$ with $\gamma(0) = z_0$ and $\gamma(1) = z$. The task at hand is to show this is a valid definition, in other words that our formula for g makes sense. Notice that we choose a particular path γ in our definition of g; what happens if we choose a different path β? Let β be another curve with $\beta(0) = z_0$ and $\beta(1) = z$. Then $\gamma - \beta$ is a closed loop in $D \backslash [-1/2, 1/2]$, which we will call Γ. The loop Γ either encloses the segment $[-1/2, 1/2]$ or is contractible in $D \backslash [-1/2, 1/2]$. If Γ is contractible, then we can apply Cauchy's theorem (because f is analytic on the region bounded by Γ), so

$$\int_\gamma f(z) dz - \int_\beta f(z) dz = \int_\Gamma f(z) dz = 0 \Rightarrow \int_\gamma f(z) dz = \int_\beta f(z) dz.$$

On the other hand, if Γ encloses the segment $[-1/2, 1/2]$ then

$$\int_\gamma f(z) dz - \int_\beta f(z) dz = \int_\Gamma f(z) dz = 2\pi i [\mathrm{Res}(f, 1/2) + \mathrm{Res}(f, 0) + \mathrm{Res}(f, -1/2)] = 0 \Rightarrow \int_\gamma f(z) dz = \int_\beta f(z) dz.$$

In either case, the integral $\int_\gamma f(z) dz$ defining g depends only on the endpoints of γ, not its actual path, and so g is a well-defined analytic function. Moreover, by the Fundamental Theorem of Calculus, $g' = f$.