Practice Problems

Math 252

March 23, 2006

1. Evaluate the following contour integrals.
   (a) $\int_{|z|=1} z^2 e^{z-2} dz$
   (b) $\int_{|z|=2} \frac{e^z}{(z-1)^3} dz$
   (c) $\int_{|z|=2} \frac{\cos z}{z-\pi/2} dz$
   (d) $\int_{|z|=1} \frac{e^z}{z(z+2)} dz$

2. Let $f(z) = \frac{z^2}{z^2+1}$.
   (a) Evaluate $\int_{|z|=1/2} f(z)dz$.
   (b) Show that $\int_{|z|=4/3} f(z)dz = \int_{|z|=5/3} f(z)dz$.
   (c) Is it true that $\int_{|z|=3/2} f(z)dz = \int_{|z|=5/2} f(z)dz$? Explain your answer.
   (d) For which positive numbers r is the contour integral $\int_{|z|=r} f(z)dz$ zero?

3. Suppose that f is analytic on the entire complex plane $\mathbb{C}$ and $\lim_{|z|\to\infty}(f(z)/z) = 0$. Show that f is constant.

4. Suppose f is analytic on the entire complex plane and $|f(z)| \leq |z|^3$ when $|z| \geq 10$.
   (a) Show that f is a polynomial of degree at most 3.
   (b) What can you say about the coefficient of the cubic term in f?

5. Find the maximum of the harmonic function $u(x, y) = e^x \cos y$ on the domain $0 \leq x \leq 1, \pi/2 \leq y \leq 3\pi/2$.

6. (a) Fix z_0 with $|z_0| < 1$, and consider
   $$T(z) = \frac{z - z_0}{1 - \bar{z}_0 z}.$$
   Show that $T(z_0) = 0$ and that T maps the unit disc $\{|z| < 1\}$ to itself. (Hint: first show that T maps the unit circle to itself and then use the maximum modulus principle.)
   (b) Let $f(z)$ be an analytic function which maps the unit disc $\{|z| < 1\}$ to itself (i.e. if $|z| < 1$ then $|f(z)| < 1$), and let $f(z_0) = w_0$. Prove the bound
   $$\left|\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}\right| \leq \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|.$$
   (Hint: look at a composition of f and functions like T.)
   (c) If the inequality you just proved is an equality, what can you say about the function f?

7. For each of the following series, determine whether it converges absolutley.
   (a) $\sum_{n=1}^{\infty} \frac{\log n}{n}$
   (b) $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{2^n}$

8. Let $\{f_n\}$ be a sequence of continuous functions defined on the interval $0 \leq t \leq 1$, and suppose $f_n \to f$ uniformly. Show that f is continuous.

9. Let $f_n(t) = \arctan(nt)$, for all real t.
   (a) Show that on the interval $\pi/2 \leq t \leq 3\pi/2$, $f_n \to \pi/2$ uniformly.
   (b) Show that, in general, the pointwise limit of $f_n(t)$ is
   $$f_n(t) \to \begin{cases} \pi/2 & t > 0 \\ 0 & t = 0 \\ -\pi/2 & t < 0. \end{cases}$$
   (c) Conclude the convergence of f_n to its pointwise limit cannot be uniform in the interval $-\pi/2 \leq t \leq \pi/2$. (Hint: if the convergence were uniform, then the limit function would be continuous.)

10. For each of the following analytic functions, determine the radius of convergence of its Taylor series centered at the origin and write out the first four terms.

   (a) $f(z) = \frac{1}{1+z}$
   (b) $f(z) = \frac{1}{\cos z}$
   (c) $f(z) = \log(1 - z)$