Solutions to the Practice Problems

Math 252

March 23, 2006

1. Evaluate the following contour integrals.

   (a) $\int_{|z|=1} z^2 e^{z-2} dz$

   The integrand $z^2 e^{z-2}$ is analytic on and inside the circle $|z| = 1$, so the contour integral is zero by Cauchy's theorem.

   (b) $\int_{|z|=2} \frac{e^z}{(z-1)^3} dz$

   Here we use the Cauchy integral formula for the derivatives of a function:

$$\int_{|z|=2} \frac{e^z}{(z-1)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2}(e^z)\bigg|_{z=1} = \frac{2\pi i e^1}{2} = \pi i e.$$

   (c) $\int_{|z|=2} \frac{\cos z}{z - \pi/2} dz$

   Notice that $|pi/2| < 2$, so we use the Cauchy integral formula:

$$\int_{|z|=2} \frac{\cos z}{z - \pi/2} dz = 2\pi i \cos(\pi/2) = 0.$$

   (d) $\int_{|z|=1} \frac{e^z}{z(z+2)} dz$

   We first do a partial fractions decomposition:

$$\frac{e^z}{z(z+2)} = e^z \left[\frac{1/2}{z} - \frac{1/2}{z+2} \right].$$

   Then

$$\int_{|z|=1} \frac{e^z}{z(z+2)} dz = \frac{1}{2} \int_{|z|=1} \left[\frac{e^z}{z} - \frac{e^z}{z+2} \right] dz = \frac{1}{2} \int_{|z|=1} \frac{e^z}{z} dz = \pi i e^0 = \pi i.$$

   We have used the fact that $e^z/(z+2)$ is analytic inside the unit disc, so that part of the contour integral vanishes.

2. Let $f(z) = \frac{z^2}{z^2+1}$.

   (a) Evaluate $\int_{|z|=1/2} f(z) dz$.

   Let's look at the function f:

$$f(z) = \frac{z^2}{z^2+1} = \frac{z^2}{(z+i)(z-i)}.$$

   This function is analytic everywhere except where the denominator is 0, which occurs at $z = \pm i$. In particular, it is analytic on and inside the circle $\{|z| = 1/2\}$, so by Cauchy's theorem, $\int_{|z|=1/2} f(z) dz = 0$.

   (b) Show that $\int_{|z|=4/3} f(z) dz = \int_{|z|=5/3} f(z) dz$.

   Note that the singularities of f lie on the circle of radius 1, so f is analytic on the region $4/3 \leq |z| \leq 5/3$. Then by the deformation theorem,

$$\int_{|z|=4/3} f(z) dz = \int_{|z|=5/3} f(z) dz.$$

   (c) Is it true that $\int_{|z|=3/2} f(z) dz = \int_{|z|=5/2} f(z) dz$? Explain your answer.

   Yes, again by the deformation theorem.

   (d) For which positive numbers r is the contour integral $\int_{|z|=r} f(z) dz$ zero?

   In fact, for any $r \neq 1$, the contour integral $\int_{|z|=r} f(z) dz = 0$. To see this, we first rewrite the function f:

$$f(z) = \frac{z^2}{z^2+1} = z^2 \left[\frac{1}{(z+i)(z-i)} \right] = z^2 \left[\frac{1/2}{z+i} - \frac{1/2}{z-i} \right].$$

   For $r < 1$, the function f is analytic inside the disc $|z| = r$, so the integral is zero by Cauchy's theorem. For $r > 1$, we can apply the Cauchy integral formula to the function $g(z) = z^2$:

$$\int_{|z|=r} \frac{z^2 dz}{z^2+1} = \frac{1}{2} \int_{|z|=r} z^2 \left[\frac{1}{z+i} - \frac{1}{z-i} \right] dz = \pi i (g(-i) - g(i)) = \pi i(-1+1) = 0.$$

3. Suppose that f is analytic on the entire complex plane $\mathbb{C}$ and $\lim_{|z|\to\infty}(f(z)/z) = 0$. Show that f is constant. Pick $z_0 \in \mathbb{C}$. By the Cauchy integral formula, if $R > |z_0|$ then

$$f'(z_0) = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{(z-z_0)^2}dz.$$

Now take the length of this integral and use our limit as $|z| \to \infty$:

$$|f'(z_0)| = \frac{1}{2\pi}\left|\int_{|z|=R} \frac{f(z)}{(z-z_0)^2}dz\right| \leq \frac{1}{2\pi}\int_{|z|=R} \frac{|f(z)|}{|z-z_0|^2}d|z| \leq \frac{1}{2\pi}\int_{|z|=R} \frac{|f(z)|}{R^2+|z_0|^2-2R|z_0|} \to 0$$

as $R \to \infty$ because $|f(z)| < R$ as $|z| \to \infty$ and $R^2 - 2R|z_0| > R/2$ for R sufficently large. As z_0 was arbitrary, we have shown that f' is zero everywhere, which implies that f is constant.

4. Suppose f is analytic on the entire complex plane and $|f(z)| \leq |z|^3$ when $|z| \geq 10$.

   (a) Show that f is a polynomial of degree at most 3.
   We will show the fourth derivative of f is zero. By the Cauchy integral formula, for any z_0 and $R > |z_0|$,

   $$|f^{(4)}(z_0)| \leq \frac{4!}{2\pi}\int_{|z|=R} \frac{|f(z)|}{|z-z_0|^5}d|z| \leq \frac{4!}{2\pi}\int_{|z|=R} \frac{R^3}{(R-|z_0|)^5}d|z| \leq \frac{4!R^4}{(R-|z_0|)^5} \to 0$$

   as $R \to \infty$. Thus $f^{(4)}$ is identically zero, which means that f is a third degree polynomial.

   (b) What can you say about the coefficient of the cubic term in f?
   Let $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$. Then $f^{(3)} = 6a_3$. On the other hand, we can use the Cauchy integral formula as before. For $R \geq 10$, we have

   $$6|a_3| = |f^{(3)}(0)| \leq \frac{6}{2\pi}\int_{|z|=R} \frac{|f(z)|}{|z|^4}d|z| \leq \frac{62\pi R^4}{2\pi R^4} = 6.$$

   Thus $|a_3| \leq 1$.

5. Find the maximum of the harmonic function $u(x, y) = e^x \cos y$ on the domain $0 \leq x \leq 1, \pi/2 \leq y \leq 3\pi/2$.
   By the maximum principle, the maximum must occur on the boundary of the rectangle. First, observe that $\cos(\pi/2) = 0 = \cos(3\pi/2)$, so

   $$u(x, \pi/2) = 0 = u(x, 3\pi/2).$$

   Next we plug in $x = 0$:

   $$u(0, y) = e^0 \cos y = \cos y.$$

   For $\pi/2 \leq y \leq 3\pi/2$, $\cos y$ reaches its maximum value of 0 at $y = \pi/2$ and $y = 3\pi/2$. (It reaches a minimum value of -1 at $y = \pi$.) Finally, we plug in $x = 1$:

   $$u(1, y) = e^1 \cos y = e \cos y.$$

   For $\pi/2 \leq y \leq 3\pi/2$, $e \cos y$ reaches its maximum value of 0 at $y = \pi/2$ and $y = 3\pi/2$. (It reaches a minimum value of $-e$ at $y = \pi$.) Thus, the maximum value of u is 0, which occurs along the edges $y = \pi/2$ and $y = 3\pi/2$. Also, the minimum value of u is $-e$, which occurs at the point $(1, \pi)$.

6. (a) Fix z_0 with $|z_0| < 1$, and consider

   $$T(z) = \frac{z - z_0}{1 - \bar{z}_0 z}.$$

   Show that $T(z_0) = 0$ and that T maps the unit disc $\{|z| < 1\}$ to itself. (Hint: first show that T maps the unit circle to itself and then use the maximum modulus principle.)
   First observe that

   $$T(z_0) == \frac{z_0 - z_0}{1 - |z_0|^2} = 0.$$

   Next, if $|z| = 1$, then

   $$|T(z)|^2 = T(z)\bar{T}(z) = \frac{z - z_0}{1 - \bar{z}_0 z} \cdot \frac{\bar{z} - \bar{z}_0}{1 - z_0 \bar{z}} = \frac{|z|^2 + |z_0|^2 - \bar{z}_0 z - z_0 \bar{z}}{1 + |z_0|^2|z|^2 - \bar{z}_0 z - z_0 \bar{z}} = 1.$$

   Finally, because T is analytic on the disc $\{|z| < 1\}$, we can apply the maximum modulus principle, which in this case says that for $|z| < 1$, $|T(z)|$ is less than the maximum on the unit circle, which in this case is 1.

(b) Let $f(z)$ be an analytic function which maps the unit disc $\{|z| < 1\}$ to itself (i.e. if $|z| < 1$ then $|f(z)| < 1$), and let $f(z_0) = w_0$. Prove the bound

$$\left|\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}\right| \leq \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|.$$

(Hint: look at a composition of f and functions like T.)

We define two functions:

$$T_1(\zeta) = \frac{z - z_0}{1 - \bar{z}_0 z}, \qquad T_2(z) = \frac{z - z_0}{1 - \bar{w}_0 z}.$$

By the previous part, we know $T_1(z_0) = 0$, $T_2(w_0) = 0$, and both functions map the unit disc $\{|z| < 1\}$ to itself. Now look at the composition

$$g(\zeta) = T_2(f(T_1^{-1}(\zeta).$$

This is a function mapping the unit disc to itself and $g(0) = 0$ (by construction), so by the Schwartz lemma $|g(\zeta)| \leq |\zeta|$. Plugging in the definitions of T_1 and T_2, this says

$$\left|\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}\right| \leq \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|.$$

(c) If the inequality you just proved is an equality, what can you say about the function f?

If

$$\left|\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}\right| \leq \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|$$

for some z with $|z| < 1$, then by the Schwartz lemma $g(\zeta) = c\zeta$ for some constant c with $|c| = 1$. This means

$$c \cdot \frac{z - z_0}{1 - \bar{z}_0 z} = \frac{f(z) - w_0}{1 - \bar{w}_0 f(z)},$$

which we can rewrite as

$$f(z) = \frac{z(c - \bar{z}_0 w_0) + w_0 - c z_0}{1 - c\bar{w}_0 z_0 + z(c\bar{w}_0 - z_0)}.$$

7. For each of the following series, determine whether it converges absolutley.

(a) $\sum_{n=1}^{\infty} \frac{\log n}{n}$

Notice that for $n \geq 3$, $\log n > 1$, so $(\log n)/n > 1/n$. However, $\sum_{n=1}^{\infty} 1/n$ diverges, which means that $\sum_{n=1}^{\infty} (\log n)/n$ also diverges.

(b) $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{2^n}$

We will use the ratio test:

$$\lim_{n \to \infty} \frac{|\cos(\pi(n+1))2^n}{|\cos(\pi n)|2^{n+1}} = \lim_{n \to \infty} \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1,$$

which means the sum converges absolutely.

8. Let $\{f_n\}$ be a sequence of continuous functions defined on the interval $0 \leq t \leq 1$, and suppose $f_n \to f$ uniformly. Show that f is continuous.

Pick some t_0 with $0 \leq t_0 \leq 1$; we will show that f is continuous at t_0. Let $\epsilon > 0$. Because $f_n \to f$ uniformly, there exists N such that for $n \geq N$

$$|f_n(t) - f(t)| < \frac{\epsilon}{3}$$

for all $t \in [0, 1]$. Now, f_n is a continuous function, so there also exists a $\delta > 0$ such that $|t - t_0| < \delta$ implies $|f_n(t) - f_n(t_0)| < \epsilon/3$. Now we have

$$\begin{aligned} |f(t) - f(t_0)| &= |f(t) - f_n(t) + f_n(t) - f_n(t_0) + f_n(t_0) - f(t_0)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(t_0)| + |f_n(t_0) - f(t_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

which means that f is continuous at t_0.

9. Let $f_n(t) = \arctan(nt)$, for all real t.

(a) Show that on the interval $\pi/2 \le t \le 3\pi/2$, $f_n \to \pi/2$ uniformly.

First note that $\arctan(x) < \pi/2$ for all real numbers x. Also,

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} > 0,$$

so arctan is an increasing function of x. Thus arctan has a horizontal asymptote at $\pi/2$, which means that for $\epsilon > 0$, there exists $X > 0$ such that if $x \ge X$ then $|\arctan(x) - \pi/2| < \epsilon$. Now take let $N = (X)(2/\pi)$. When $n \ge N$, $nx > X$ on the inteval $\pi/2 \le x \le 3\pi/2$, so $|\arctan(nx) - \pi/2| < \epsilon$ for all $x \in [\pi/2, 3\pi/2]$.

(b) Show that, in general, the pointwise limit of $f_n(t)$ is

$$f_n(t) \to \begin{cases} \pi/2 & t > 0 \\ 0 & t = 0 \\ -\pi/2 & t < 0. \end{cases}$$

For $x > 0$, $\arctan(x)$ has a horizontal asymptote of $\pi/2$, while for for $x < 0$, $\arctan(x)$ has a horizontal asymptote of $-\pi/2$. This means that for any $t > 0$, as $n \to \infty$, $\arctan(nt) \to \pi/$. Meanwhile, for any $t < 0$, as $n \to \infty$, $\arctan(nt) \to -\pi/2$. Finally, no matter what n is we always have $\arctan(n \cdot 0) = \arctan(0) = 0$.

(c) Conclude the convergence of f_n to its pointwise limit cannot be uniform in the interval $-\pi/2 \le t \le \pi/2$. (Hint: if the convergence were uniform, then the limit function would be continuous.)

Note that each function $f_n(t) = \arctan(nt)$ is continuous. Thus, if the pointwise limit were a uniform limit, the limit function would also have to be continuous. However, the limit is not continuous, so the convergence cannot be uniform.

10. For each of the following analytic functions, determine the radius of convergence of its Taylor series centered at the origin and write out the first four terms.

(a) $f(z) = \frac{1}{1+z}$

This function is analytic so long as $z \ne -1$, so the radius of convergence is 1. (The largest disc centered at the origin on which f is analytic is the disc of radius 1.) To compute the start of the Taylor series:

$$f(0) = 1, \quad f'(0) = \left.\frac{-1}{(1+z)^2}\right|_{z=0} = -1, \quad f''(0) = \left.\frac{1}{(1+z)^3}\right|_{z=0} = 1, \quad f'''(0) = \left.\frac{-1}{(1+z)^4}\right|_{z=0} = -1.$$

Thus $f(z) = 1 - z + z^2/(2!) - z^3/(3!) + \ldots$

(b) $f(z) = \frac{1}{\cos z}$

The largest disc centered at the origin on which f is analytic is the disc of radius $\pi/2$ (because $\cos(\pi/2) = 0$), so the radius of convergence of f is $\pi/2$. We again compute some derivatives:

$$f(0) = 1, \quad f'(0) = \frac{\sin(0)}{\cos^2(0)} = 0, \quad f''(0) = \frac{1 + 2\sin^2(0)}{\cos^3(0)} = 1, \quad f'''(0) = \frac{7\sin(0)}{\cos^4(0)} = 0.$$

Thus $f(z) = 1 + z^2/(2!) + \ldots$

(c) $f(z) = \log(1 - z)$

This function is analytic so long as $1 - z$ is not either 0 or a negative real number, i.e. so long as z is not a real number bigger than or equal to 1. (We've chosen the principle branch of log here, which gives us the largest radius of convergence.) Thus the largest disc centered at the origin on which f is analytic is the disc of radius 1, so the radius of convergence is 1. Computing some derivatives:

$$f(0) = \log(1) = 0, \quad f'(0) = -\frac{1}{1-0} = -1, \quad f''(0) = \frac{1}{(1-0)^2} = 1, \quad f'''(0) = -\frac{1}{(1-0)^3} = -1.$$

Thus $f(z) = -z + z^2/(2!) - z^3/(3!) + \ldots$