

Solutions to the Practice Problems
Math 252
February 13, 2006

1. Rewrite each of the following complex numbers in the form $z = a + ib$, where a and b are real numbers

(a) $z = \sqrt{1+i}$

We first write things in polar form:

$$1+i = \sqrt{2}(\cos(\pi/4)+i\sin(\pi/4)) \Rightarrow \sqrt{1+i} = (2)^{1/4}(\cos(\pi/8)+i\sin(\pi/8)) = (2)^{1/4}\cos(\pi/8)+i(2)^{1/4}\sin(\pi/8).$$

(b) $z = e^{3+i\pi/2}$

$$e^{3+i\pi/2} = e^3 \cdot e^{i\pi/2} = e^3(\cos(\pi/2) + i\sin(\pi/2)) = ie^3$$

(c) $z = [\cos(\pi/3) + i\sin(\pi/3)]^{1/3}$

$$[\cos(\pi/3) + i\sin(\pi/3)]^{1/3} = \cos(\pi/9) + i\sin(\pi/9)$$

2. Rewrite each of the following complex numbers in polar form $z = re^{i\theta}$, where $r \geq 0$ and $-\pi < \theta \leq \pi$.

(a) $1 - i$

$$|1 - i| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2} \text{ and } \arg(1 - i) = \arctan(-1) = 3\pi/4, \text{ so } 1 - i = \sqrt{2}e^{3i\pi/4}.$$

(b) $[-\sqrt{3} + i]^{1/3}$

First notice that $|-\sqrt{3} + i| = \sqrt{3+1} = 2$ and $\arg(-\sqrt{3} + i) = \arctan(-1/\sqrt{3}) = 5\pi/6$. Thus

$$[-\sqrt{3} + i]^{1/3} = 2^{1/3}e^{5i\pi/18}.$$

(c) $\sqrt{1 + e^{-i\pi/2}}$

First, $e^{-i\pi/2} = \cos(-\pi/2) + i\sin(-\pi/2) = -i$, so

$$\sqrt{1 + e^{-i\pi/2}} = \sqrt{1 - i} = \sqrt{\sqrt{2}e^{-i\pi/4}} = (2)^{1/4}e^{-i\pi/8}.$$

3. Describe all the z which solve each of the following equations.

(a) $\log(1 + z) = i$

We have

$$i = \log(1 + z) = \log|1 + z| + i\arg(1 + z).$$

Separating out the real and imaginary parts,

$$|1 + z| = e^0 = 1, \quad \arg(1 + z) = 1,$$

so we can write $z = -1 + e^{i+2n\pi}$, for any integer n .

(b) $e^{z^2} = -2$

Let $w = u + iv = z^2$, so

$$e^w = e^u(\cos v + i\sin v) = -2 \Rightarrow e^u = 2, v = (2n + 1)\pi,$$

where n can be any integer. Then

$$z^2 = (x + iy)^2 = u + iv = \log(2) + (2n + 1)i\pi = r_n e^{i\theta_n},$$

where

$$r_n^2 = (\log 2)^2 + (2n + 1)^2\pi^2, \quad \theta_n = \arctan\left(\frac{(2n + 1)\pi}{\log 2}\right).$$

Then $z = r_n^{1/2}e^{i\theta_n/2}$, for any integer n .

(c) $\sqrt{2z+1} = 1 + i$

$$2z + 1 = (1 + i)^2 = 1 + 2i - 1 = 2i \Rightarrow z = -\frac{1}{2}(1 + 2i)$$

4. For each of the following functions $f(z)$ and domains D , describe the image domain $f(D)$.

- (a) $f = z^3$, $D = \{z \mid 1 \leq |z| \leq 2, \pi/2 \leq \arg(z) \leq \pi\}$

Recall that $f(z) = z^3$ acts to triple angles and cube lengths. Thus the image of the domain D is the domain

$$\{z \mid 1 \leq |z| \leq 8, 3\pi/2 \leq \arg(z) \leq 3\pi\} = \{z \mid 1 \leq |z| \leq 8, -\pi/2 \leq \arg(z) \leq \pi\}.$$

- (b) $f = \sin z$, $D = \{z = x + iy \mid 0 \leq x \leq \pi, y > 0\}$

First we unravel $\sin z$:

$$\begin{aligned} \sin(x + iy) &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x)) \\ &= \frac{1}{2} \sin x (e^y + e^{-y}) + \frac{i}{2} \cos x (e^y - e^{-y}) = \sin x \cosh y + i \cos x \sinh y. \end{aligned}$$

Since $\sinh y > 0$ for $y > 0$ and $\cos x$ takes all values between 1 and -1 for $0 \leq x \leq \pi$, the imaginary part can be anything. On the other hand, for $y > 0$ we have $\cosh y > 1$ and for $0 \leq x \leq \pi$ we have $0 \leq \sin x \leq 1$, so the real part must be positive. Thus the image of D is the right half plane $\{x + iy \mid x > 0\}$.

5. Determine whether each of the functions below is analytic at the given point. Be sure to explain your answer.

- (a) $f(z) = \frac{1+z}{1-z}$, $z_0 = 2$

f is a rational function, so it's analytic at z_0 so long as the denominator is nonzero there. In this case, $1 - z_0 = -1 \neq 0$, so yes, f is analytic at $z_0 = 2$.

- (b) $f(z) = |z|^2$, $z_0 = i$

No. Let's check the Cauchy Riemann equations:

$$u = x^2 + y^2, \quad v = 0, \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 0.$$

The two derivatives are not equal, f is not analytic.

- (c) $f(z) = u(x, y) + iv(x, y)$ where $u = x^2 - y^2$, $v = -2xy$, $z_0 = 0$

No. Again, we can check the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2x.$$

These are not equal, so the function is not analytic.

- (d) $f(z) = \log(1 + z)$, $z_0 = 1$

We really need to pick a branch of the logarithm to have this function make sense, so let's pick the principal branch, with $-\pi < \Im[\log(z)] < \pi$. Then $\log(\cdot)$ is analytic so long as the thing inside the logarithm stays away from the negative real axis. In this case, $1 + z_0 = 1$, so the function is analytic.

6. Let $f(x + iy) = u(x, y) + iv(x, y)$ be an analytic function.

- (a) Write out the complex derivative of f at $z_0 = 0$ in terms of partial derivatives of u and v , approaching along the ray $x = y > 0$.

We have

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0^+} \left[\frac{u(x, x) + iv(x, x) - u(0, 0) - iv(0, 0)}{x + ix} \right] \\ &= \lim_{x \rightarrow 0^+} \left[\frac{(x - ix)(u(x, x) + iv(x, x) - u(0, 0) - iv(0, 0))}{(x + ix)(x - ix)} \right] \\ &= \lim_{x \rightarrow 0^+} \left[\frac{x(u(x, x) - u(0, 0) + v(x, x) - v(0, 0)) + ix(-u(x, x) + u(0, 0) + v(x, x) - v(0, 0))}{2x^2} \right] \\ &= \lim_{x \rightarrow 0^+} \left[\frac{u(x, x) - u(0, 0) + v(x, x) - v(0, 0)}{2x} \right] + i \lim_{x \rightarrow 0^+} \left[\frac{-u(x, x) + u(0, 0) + v(x, x) - v(0, 0)}{2x} \right] \\ &= \frac{1}{2} \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right] + \frac{i}{2} \left[-\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right], \end{aligned}$$

where all the partial derivatives are evaluated at $(0, 0)$.

- (b) Do the same thing approaching along the ray $x = -y > 0$.

This is very similar:

$$\begin{aligned}
 f'(0) &= \lim_{x \rightarrow 0^+} \left[\frac{u(x, -x) + iv(x, -x) - u(0, 0) - iv(0, 0)}{x - ix} \right] \\
 &= \lim_{x \rightarrow 0^+} \left[\frac{(x + ix)(u(x, -x) + iv(x, -x) - u(0, 0) - iv(0, 0))}{(x + ix)(x - ix)} \right] \\
 &= \lim_{x \rightarrow 0^+} \left[\frac{x(u(x, -x) - u(0, 0) - v(x, -x) + v(0, 0)) + ix(u(x, -x) - u(0, 0) + v(x, -x) - v(0, 0))}{2x^2} \right] \\
 &= \lim_{x \rightarrow 0^+} \left[\frac{u(x, -x) - u(0, 0) - v(x, -x) + v(0, 0)}{2x} \right] + i \lim_{x \rightarrow 0^+} \left[\frac{u(x, -x) - u(0, 0) + v(x, -x) - v(0, 0)}{2x} \right] \\
 &= \frac{1}{2} \left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right] + \frac{i}{2} \left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right],
 \end{aligned}$$

where all the partial derivatives are evaluated at $(0, 0)$.

- (c) Write out the resulting differential equations u and v must satisfy.

We equate the real and imaginary parts of the two computations above:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \Leftrightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

These are the familiar Cauchy Riemann equations.

In fact, it is quite remarkable that you **only** get the Cauchy Riemann equations when comparing the limits along different rays of the difference quotients for f' . The general computation work in much the same way.

7. Consider the function $f(z) = z^2 - z$.

- (a) Describe the infinitesimal behavior of f near $z_0 = 1 - i$.

Notice that $f'(z) = 2z$, so $f'(1 - i) = 2 - 2i$. Thus the infinitesimal behavior of f near $1 - i$ is to dilate by $|2 - 2i| = 2\sqrt{2}$ and rotate by $\arctan(-2/2) = -\pi/4$.

- (b) This function f is conformal near all points except one point z_1 . Find this point.

$f(z) = z^2$ is conformal except where $0 = f'(z) = 2z$, which happens when $z_1 = 0$.

- (c) Describe the infinitesimal behavior of f near this point z_1 .

Now the derivative is zero, so we have to look at the next term in the expansion: $f(z) = f(0) + zf'(0) + \frac{1}{2}z^2f''(0) + \dots$. The next term is z^2 , which means that f acts infinitesimally to double angles and square lengths near $z_1 = 0$.

8. Evaluate the following contour integrals.

- (a) $\int_{\gamma} (1 + 3z^2)dz$, $\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$

We use the fundamental theorem of calculus, and the fact that

$$\frac{d}{dz}(z + z^3) = 1 + 3z^2.$$

Also notice that $\gamma(\pi) = -1$ and $\gamma(0) = 1$. Then we have

$$\int_{\gamma} (1 + 3z^2)dz = (z + z^3)|_{z=-1} - (z + z^3)|_{z=1} = -4.$$

- (b) $\int_{|z|=1} \frac{dz}{z}$

This time we actually write out the contour integral. First,

$$f(z) = u(x, y) + iv(x, y) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2},$$

so

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.$$

We can parameterize γ as $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$, so then

$$\begin{aligned} \int_{\gamma} \frac{dz}{z} &= \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy = \int_0^{2\pi} (ux' - vy') dt + i \int_0^{2\pi} (vx' + uy') dt \\ &= \int_0^{2\pi} (\cos t)(-\sin t) - (-\sin t)(\cos t) dt + i \int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t) dt \\ &= i \int_0^{2\pi} dt = 2\pi i. \end{aligned}$$

(c) $\int_{|z-3|=1} z^2 e^{3z} dz$

This is zero by Cauchy's theorem.