Solutions to the Midterm Exam
Math 252
April 5, 2006

1. Evaluate the following contour integrals. (Think before you compute.)

   (a) (5 points) $\int_{|z|=1} e^{z^3}\cos(z+1)dz$

   The integrand $e^{z^3}\cos(z+1)$ is analytic everywhere, so by Cauchy's theorem the contour integral is zero.

   (b) (5 points) $\int_{|z|=1} \frac{\cos(z)}{z}dz$

   This is a Cauchy-type integral. By the Cauchy integral formula

$$\int_{|z|=1} \frac{\cos z}{z}dz = 2\pi i\cos(0) = 2\pi i.$$

2. Consider the function

$$f(z) = \frac{z^3}{z^2-1}.$$

   (a) (5 points) Show that $\int_{|z|=1/2} f(z)dz = 0$.

   The function f is analytic except at $z = \pm 1$, which is where the denominator is zero. In particular, f is analytic inside and on the circle of radius $1/2$. Thus, by Cauchy's theorem,

$$\int_{|z|=1/2} f(z)dz = 0.$$

   (b) (5 points) Is it true that $\int_{|z|=3/2} f(z)dz = \int_{|z|=5/2} f(z)dz$? Explain your answer.

   f is analytic on the region between $|z| = 3/2$ and $|z| = 5/2$, so by the deformation theorem the two contour integrals are equal. Another way to see this is to connect the two circles $|z| = 3/2$ and $|z| = 5/2$ along the real axis, using the curve $\beta = \{(x,0) \mid 3/2 \leq x \leq 5/2\}$. Then by Cauchy's theorem

$$0 = \int_{|z|=5/2} f(z)dz - \int_\beta f(z)dz - \int_{|z|=3/2} f(z)dz + \int_\beta f(z)dz = \int_{|z|=5/2} f(z)dz - \int_{|z|=3/2} f(z)dz.$$

   (c) (5 points) Evaluate $\int_{|z|=4/3} f(z)dz$.

   We will first use partial fractions to rewrite f:

$$f(z) = \frac{z^3}{z^2-1} = z^3\left[\frac{1}{(z-1)(z_1)}\right] = \frac{z^3}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right].$$

   Then

$$\int_{|z|=4/3} f(z)dz = \frac{1}{2}\int_{|z|=4/3} z^3\left[\frac{1}{z-1} - \frac{1}{z+1}\right] = \pi i(1)^3 - \pi i(-1)^3 = 2\pi i.$$

3. Let $f(z)$ be a function which is analytic on the entire complex plane, and suppose $|f(z)| \leq 3|z|^2$ for $|z| \geq 100$.

   (a) (5 points) Prove that f is a polynomial of degree at most 2.

   We will show that $f'''(z_0) = 0$ for any fixed z_0, which implies that f is a polynomial of degree at most 2. By the Cauchy integral formula, for $R > |z_0|$

$$\begin{aligned} |f'''(z_0)| &= \left|\frac{3!}{2\pi i}\int_{|z|=R} \frac{f(z)}{(z-z_0)^4}dz\right| \leq \frac{6}{2\pi}\int_{|z|=R} \frac{|f(z)|}{|z-z_0|^4}|dz| \\ &\leq \frac{6}{2\pi}\int_{|z|=R} \frac{3|z|^2}{(|z|-|z_0|)^4}|dz| = \frac{18}{2\pi}\int_{|z|=R} \frac{R^2}{(R-|z_0|)^4}|dz| \\ &= \frac{18R^2}{2\pi(R-|z_0|)^4}\int_{|z|=R} |dz| = \frac{18R^3}{(R-|z_0|)^4} \to 0 \end{aligned}$$

   as $R \to \infty$. Thus f''' is identically zero, and so f is a polynomial of degree at most 2.

(b) (5 points) Since f is a quadratic polynomial, one can write it as $f(z) = a_2 z^2 + a_1 z + a_0$. What can you say about a_2?

Differentiate $a_2 z^2 + a_1 z + a_0$ twice to get $f'' = 2a_2$. On the other hand, we have the Cauchy integral formula. So

$$2|a_2| = |f''(0)| \le \frac{1}{\pi} \int_{|z|=R} \frac{|f(z)|}{|z|^3} |dz| \le \frac{1}{\pi} \int_{|z|=R} \frac{3R^2}{R^3} |dz| = \frac{12\pi R^3}{2\pi R^3} = 6.$$

Thust $|a_2| \le 3$.

4. (5 points) Find the maximum of the harmonic function $u(x, y) = x^2 - y^2$ on the domain $\{x + iy \mid -1 \le x \le 1, -\pi/2 \le y \le \pi/2\}$.

By the maximum principle, the maximum of u occurs on the boundary of the rectangle, so we'll test each side in turn. When $x = \pm 1$, $u(\pm 1, y) = 1 - y^2$, which has a maximum value of 1 when $y = 0$. This takes care of the left and right sides. For the top and bottom sides, we set $y = \pm\pi/2$, where $u(x, \pm\pi/2) = x^2 - \pi^2/4$. This has a maximum value of $1 - \pi^2/4 < 1$, which occurs at $x = \pm 1$. Thus, over the while rectangle, the maximum value is 1, which occurs at $(\pm 1, 0)$.

5. Let $f(z) = \frac{z}{z^2-1}$.

(a) (5 points) What is the radius of convergence of the Taylor series for f, centered at the point $z_0 = 2$?

The function $f(z)$ is analytic except at $z = \pm 1$. Thus largest disc centered at $z_0 = 2$ on which f is analytic is a disc of radius 1, so the radius of convergence is 1.

(b) (5 points) Compute the first four coefficients of this series.

We have $a_n = f^{(n)}(2)/n!$. First we compute some derivatives:

$$\begin{aligned}
f'(z) &= \frac{1}{z^2-1} - \frac{2z^2}{(z^2-1)^2} = \frac{z^2-1-2z^2}{(z^2-1)^2} = \frac{-1-z^2}{(z^2-1)^2} \\
f''(z) &= \frac{-2z}{(z^2-1)^2} + \frac{2(1+z^2)(2z)}{(z^2-1)^3} = \frac{-2z(z^2-1)+4z(z^2+1)}{(z^2-1)^3} = \frac{2z^3+6z}{(z^2-1)^2} \\
f'''(z) &= \frac{6z^2+6}{(z^2-1)^3} + \frac{(2z^3+6z)(-3)(2z)}{(z^2-1)^4} = \frac{(6z^2+6)(z^2-1)-6z(2z^3+6z)}{(z^2-1)^4} = \frac{-6z^4-36z^2-6}{(z^2-1)^4}.
\end{aligned}$$

Now we plug in $z_0 = 2$ to see

$$f(2) = \frac{2}{3}, \quad f'(2) = -\frac{5}{9}, \quad \frac{f''(2)}{2!} = \frac{14}{27}, \quad \frac{f'''(2)}{3!} = \frac{-46}{81}$$