Solutions to the First Midterm Exam

Math 252

Feb. 17, 2005

1. (a) (5 points) Rewrite $\sqrt{4e^{2\pi i/3}}$ in the form $a + ib$, where a and b are real numbers.

   Recall that taking a square root will halve angles and take the square root of lenghts, so

$$\sqrt{4e^{2\pi i/3}} = \pm 2e^{\pi i/3} = \pm 2(\cos(\pi/3) + i\sin(\pi/3)) = \pm 2(1/2 + i\sqrt{3}/2) = \pm(1 + i\sqrt{3}).$$

   (b) (5 points) Write $(1 - i)^2$ in polar form as $re^{i\theta}$, where $r > 0$ and θ is a real number.

   First compute lengths:

$$|(1 - i)^2| = |1 - i|^2 = 1 + 1 = 2.$$

   The angle of $1-i$ is $\arg(1-i) = -\pi/4$, so the angle of $(1-i)^2$ is $2(-\pi/4) = -\pi/2$. Thus $(1-i)^2 = 2e^{-i\pi/2}$.

2. Find all the complex numbers z which solve the following equations.

   (a) (5 points) $e^{2z+1} = -e$

   Let $z = x + iy$, so $2z + 1 = 2x + 1 + 2iy$, so

$$e^{2z+1} = e^{2x+1+2iy} = e^{2x+1}(\cos y + i \sin y) = -e.$$

   This means $2x + 1 = 1$, so $x = 0$. Also, $2y = (2n + 1)\pi$ for any integer n, so $y = (2n + 1)\pi/2$. Putting this together, we have

$$z = \frac{(2n + 1)i\pi}{2}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

   (b) (5 points) $(z + 2)^2 = -2$

   Taking the square root: $z + 2 = \pm i$, so $z = -2 \pm i$.

3. (10 points) Let $f(z) = e^{-z}$ and $D = \{z = x + iy \mid x \geq 0, -\pi/2 \leq y \leq \pi/2\}$. Describe and sketch the image domain $f(D)$.

   Let $z = x + iy$, so then $e^{-z} = e^{-x-iy} = e^{-x}(\cos y - i \sin y)$. This means

$$|f(z)| = e^{-x}, \qquad \arg(f(z)) = -y.$$

   We're given that $x \geq 0$ and $-\pi/2 \leq y \leq \pi/2$, so $0 < |f(z)| \leq 1$ and $-\pi/2 \leq \arg(f(z)) \leq \pi/2$. Thus the image domain $f(D)$ is the closed right half-disc of radius 1, centered at the origin, minus the origin.

4. Let $f(z) = z^3 - 3z$.

   (a) (5 points) Describe the infintesimal behavior of f near the point $z_0 = i$.

   The infinitesimal behavir of f at $z = i$ is determined by the derivative $f'(i) = [3z^2 - 3]\big|_{z=i} = -6$. Thus, infinitesimally, f dilates by $|f'(i)| = |-6| = 6$ and rotates by $\arg(f'(i)) = \arg(-6) = \pi$.

   (b) (5 points) The function f is conformal everywhere except at two points z_1 and z_2. Find these points.

   f will be conformal at z_0 provided $f'(z_0) \neq 0$, so to find the ponts where f is not conformal, we need to find where $f' = 0$. We have

$$0 = f'(x) = 3z^2 - 3 = 3(z^2 - 1) \Leftrightarrow z = \pm 1.$$

   So the non-conformal points are $z_1 = 1$ and $z_2 = -1$.

5. Consider the function $f(z) = \bar{z}$.

   (a) (5 points) Show that f is not analytic anywhere.

   If we write $f$in real an imaginary parts are $f = u + iv$, then $u(x, y) = x$ and $v(x, y) = -y$. Thus

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial y} = -1,$$

   which means u do not satisfy the Cauchy-Riemann equations and f cannot be analytic.

(b) (5 points) If $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi$, set up **but do not evaluate** the contour integral

$$\int_\gamma f(z)dz.$$

Along γ, we have $x = \cos t$, $x' = -\sin t$, $y = \sin t$, $y' = \cos t$, $u = x = \cos t$, and $v = -y = -\sin t$. Putting this all together, we get

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma udx - vdy + i\int_\gamma vdx + udy \\
&= \int_0^\pi (\cos t)(-\sin t) - (-\sin t)(\cos t)dt + i\int_0^\pi (-\sin t)(-\sin t) + (\cos t)(\cos t)dt \\
&= i\int_0^\pi 1dt = \pi i.
\end{aligned}$$