

The Laplacian in polar coordinates

There are many ways to derive the proper formula for the Laplacian in polar coordinates, but here's one. You might try this yourself and only refer to this computation if you get lost. Anyhow, it's good to sit down and compute this at least once in your life.

We will start with

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$

Below we will let u be any function of two variables, with at least two derivatives. Then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = \cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial u}{\partial y}.$$

Taking one more derivative, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r} \left[\cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial u}{\partial y} \right] \\ &= \cos \theta \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial r} \right] + \sin \theta \left[\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial r} \right] \\ &= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2} \end{aligned}$$

Similarly,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial u}{\partial x} + r \cos \theta \frac{\partial u}{\partial y}.$$

Taking one more derivative, we see

$$\begin{aligned} \frac{\partial^2 u}{\partial \theta^2} &= r \frac{\partial}{\partial \theta} \left[-\sin \theta \frac{\partial u}{\partial x} + \cos \theta \frac{\partial u}{\partial y} \right] \\ &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r \left[-\sin \theta \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \theta} \right) + \cos \theta \left(\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \theta} \right) \right] \\ &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r \left[-\sin \theta \left(-r \sin \theta \frac{\partial^2 u}{\partial x^2} + r \cos \theta \frac{\partial^2 u}{\partial x \partial y} \right) + \cos \theta \left(-r \sin \theta \frac{\partial^2 u}{\partial x \partial y} + r \cos \theta \frac{\partial^2 u}{\partial y^2} \right) \right] \\ &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r^2 \left[\sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \right] \\ &= -r \frac{\partial u}{\partial r} + r^2 \left[\sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \right] \end{aligned}$$

Now we're ready to put everything together:

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2} - \frac{1}{r} \frac{\partial u}{\partial r} + \sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \\ &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 u}{\partial x^2} + (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 u}{\partial y^2} - \frac{1}{r} \frac{\partial u}{\partial r} \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r} \frac{\partial u}{\partial r}, \end{aligned}$$

which we can rearrange to read

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u.$$