Practice Problems

Math 2280

March 20, 2004

1. Consider the differential equation
$$x' + 2tx = t^2.$$
   (a) Find the general solution. (Hint: use an integrating factor.)
   (b) Find the solution to the initial value problem with $x(0) = -1$.

2. Consider the differential equation $x' = (1 + 2t)(x^2 - 1)$.
   (a) Find all the fixed points (i.e. constant solutions).
   (b) Find the general solution. (Hint: separate variables.)

3. Consider the differential equation $x' = (e^x - 1)(1 - x^2)$.
   (a) Find all the fixed points (constant solutions).
   (b) Sketch the phase portrait of this differential equation.
   (c) Classify which of these fixed points is stable.

4. Consider the differential equation $x'' - 5x' + 4x = 0$.
   (a) Verify that $x_1 = e^{4t} + e^t$, $x_2 = e^{4t} - e^t$ and $x_3 = 2e^{4t} - 2e^t$ are all solutions to the equation.
   (b) Do $\{x_1, x_2\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)
   (c) Do $\{x_1, x_3\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)

5. Consider a general second order linear differential equation of the form $x'' + p(t)x' + q(t)x = 0$.
   (a) Prove that the space of solutions is a two-dimensional vector space.
   (b) Prove that if x_1 and x_2 are solutions and they are linearly independent at $t = 0$ (i.e. one can write the solution to any initial value problem with $x(0) = a$ and $x'(0) = b$ as a linear combination of x_1 and x_2) then x_1 and x_2 are linearly independent for all t.

6. Consider the differential equation
$$x'' - 6x' + 9x = \sin t.$$
   (a) Find the general solution to the associated homogeneous problem.
   (b) Find a solution to the inhomogeneous problem using your favorite method.
   (c) Find the solution to the inhomogeneous initial value problem with $x(0) = 1$ and $x'(0) = -1$.

7. Consider the differential equation
$$x'' - 2x' + x = \frac{1}{1 + e^t}.$$
   (a) Find the general solution to the associated homogeneous problem.
   (b) Find a solution to the inhomogeneous problem using your favorite method.
   (c) Find the solution to the inhomogeneous initial value problem with $x(0) = 2$ and $x'(0) = 0$.

8. Consider the system of differential equations
$$x' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} x.$$
   (a) Find the general solution using your favorite method.
   (b) Find the solution to the initial value problem with $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   (c) What is the stability behavior of the fixed point $x = 0$? Explain your answer.

9. Consider the one-parameter family of differential equations with parameter α given by
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_1 - \alpha x_1 x_2 \\ x_2 + \alpha x_1 x_2 \end{pmatrix} = F(x_1, x_2).$$

(a) Verify that the only fixed points of this system are $(0,0)$ and $(-1/\alpha, 1/\alpha)$ for $\alpha \neq 0$, and only $(0,0)$ for $\alpha = 0$.

(b) Linearize this system about each fixed point.

(c) What is the stability behavior of each fixed point? Explain your answer.

(d) Draw some representative phase portraits.

(e) Is there a bifurcation point in α? If there is, find it. Explain your answer.

10. Consider the one-parameter family of differential equations with parameter α given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} (e^{x_1} - 1)(\alpha - x_2) \\ (e^{x_2} - 1)(\alpha - x_1) \end{pmatrix} = F(x_1, x_2).$$

(a) Verify that the only fixed points of this system are $(0,0)$ and (α, α).

(b) Linearize this system about each fixed point.

(c) What is the stability behavior of each fixed point? Explain your answer.

(d) Draw some representative phase portraits.

(e) Is there a bifurcation point in α? If there is, find it. Explain your answer.

11. Let $f(x) = e^{x^2}$ for $-\pi \leq x \leq \pi$

(a) Is f an even function, an odd function, or neither?

(b) **Without computing them,** what can you say about the Fourier coefficients b_k in the Fourier series for f?

12. Let f be an odd function on $[-\pi, \pi]$.

(a) What is the average value of f?

(b) What can you say about the Fourier coefficients of f?

(c) Suppose that in addition f is π-periodic. Can you say anything more about the Fourier series of f?

13. For which of the PDEs listed below does the principle of superposition hold (i.e. when can you add solutions to obtain another solution)?

(a) $u_{tt} = u_{xt}$

(b) $u_{tt} = u_x - u_t$

(c) $u_t^2 = u_{xx}$

(d) $u_{tx} = u_{xx} - u_{tt}$

14. Let f and g be the 2π-periodic functions defined in the interval $[-\pi, \pi]$ by

$$f(t) = \begin{cases} \pi + t & -\pi \leq t \leq 0 \\ \pi - t & 0 \leq t \leq \pi \end{cases}$$

and

$$g(t) = \begin{cases} 1 & -\pi \leq t \leq 0 \\ -1 & 0 \leq t \leq \pi. \end{cases}$$

(a) Find the Fourier series for f.

(b) Find the Fourier series for g.

(c) Compute the derivative of f formally by differentiating its Fourier series term by term. You should obtain the Fourier series for g. Use this series expansion to verify that f is piecewise C^1 (i.e. has one piecewise continuous derivative). The key point is that the formal derivative you compute by differentiating the Fourier series term by term does actually converge.

(d) Formally differentiate f once more by differentiating the Fourier series (again) term by term. Does the series you obtain converge? Explain your answer.

15. (a) Find the Fourier series expansion for the odd extension of $f(t)$ where $f(t) = \pi t - t^2$ for $0 \leq t \leq \pi$.

(b) Find the Fourier series expansion for the even extension of $f(t)$ where $f(t) = \pi t - t^2$ for $0 \leq t \leq \pi$.

(c) Find the solution to the boundary value problem

$$u_t = u_{xx}$$

for $0 \leq x \leq \pi$ and $t \geq 0$ with the initial conditions $u(x,0) = \pi x - x^2$ and boundar conditions $u(0,t) = 0 = u(\pi,t)$. (Hint: separate variables and expand in Fourier series.)

(d) Find the solution to the boundary value problem

$$u_t = u_{xx}$$

for $0 \leq x \leq \pi$ and $t \geq 0$ with the initial conditions $u(x,0) = \pi x - x^2$ and coundary conditions $u_x(0,t) = 0 = u_x(\pi,t)$. (Hint: separate variables and expand in Fourier series.)

16. (a) Find the Fourier series expansion for the odd extension of $f(t)$ where

$$f(t) = \begin{cases} t & 0 \leq t \leq \frac{\pi}{2} \\ \pi - t & \frac{\pi}{2} \leq t \leq \pi \end{cases}$$

for $0 \leq t \leq \pi$.

(b) Solve the boundary value problem

$$u_{tt} = u_{xx}$$

for $0 \leq x \leq \pi$ and $t \geq 0$ with the initial conditions $u(x,0) = f(x)$ and boundary conditions $u(0,t) = 0 = u(\pi,t)$. (Hint: separate variables and expand in Fourier series.)

(c) Write the solution you found in the previous part as the sum of a left-traveling wave and a right-traveling wave.

17. Consider the wave equation

$$\partial_t^2 u = \partial_x^2 u, \qquad u(0,t) = 0 = u(1,t),$$

where the intial velocity $\partial_t u(x,0)$ is zero and the initial position is given by

$$u(x,0) = \begin{cases} 0 & 0 \leq x < 1/4 \\ 1 & 1/4 \leq x \leq 3/4 \\ 0 & 3/4 < x \leq 1. \end{cases}$$

What is the smallest positive time t such that $u(7/8,t) \neq 0$? (Hint: use d'Alembert's solution.)

18. Solve the following heat equations.

(a) $u_t = u_{xx}$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = 2\sin(2x) - \sin(3x)$.

(b) $u_t = u_{xx}$, $u(0,t) = 0 = u(1,t)$, $u(x,0) = 1/2 - |x - 1/2|$.

(c) $u_t = uxx$, $u_x(0,t) = 0 = u_x(1,t)$, $u(x,0) = -\sin(2\pi x) + 2\sin(\pi x)$.

19. (a) Consider the heat equation

$$\partial_t u = \partial_x^2 u, \qquad u(0,t) = 0 = u(L,t), \qquad u(x,0) = f(x)$$

and define the energy at time t as

$$E(t) := \int_0^L u^2(x,t)dx.$$

Show that the energy is a decreasing function in time. Does this make sense physically? (Hint: how can you characterize decreasing functions of one variable? You might want to use the equation and the boundary conditions, and integrate by parts at some point.)

(b) Consider the same heat equation as in the previous part, but this time with the boundary conditions

$$\partial_x u(0,t) = 0 = \partial_x(L,t).$$

Is the energy still decreasing? Explain your answer.

20. The equation of motion for a damped vibrating string is

$$\partial_t^2 u + k\partial_t u = c^2 \partial_x^2 u.$$

As in the case of a free vibrating string, we will consider the case where the endpoints of the string are fixed at the same height, the length of the string is L, and look for a solution of the form

$$u(x,t) = v(x)w(t).$$

(a) What are the ODEs which v and w satisfy?

(b) What boundary conditions and/or initial conditions must v and w satisfy?

(c) Can you use these boundary/initial conditions to say anything about the solutions to the ODEs?