Solutions to the Practice Problems

Math 2280

1. Consider the differential equation
$$x' + 2tx = t^2.$$
   (a) Find the general solution. (Hint: use an integrating factor.)
   Notice that if we multiply the equation by e^{t^2}, we have
$$t^2 e^{t^2} = x' e^{t^2} + x \cdot 2te^{t^2} = \frac{d}{dt}(xe^{t^2}).$$
   Thus we have (after integrating with respect to t, using the chain rule and solving)
$$x(t) = e^{-t^2}[\int_0^t s^2 e^{s^2} ds + c] = e^{-t^2}[(t/2)e^{t^2} - (1/2)\int_0^t e^{s^2} ds + c]$$
   for some constant c.

   (b) Find the solution to the initial value problem with $x(0) = -1$.
   We use the general solution above and evaluate at $t = 0$. We have
$$-1 = x(0) = c,$$
   so
$$x(t) = e^{-t^2}[(t/2)e^{t^2} - (1/2)\int_0^t e^{s^2} ds - 1].$$

2. Consider the differential equation $x' = (1+2t)(x^2-1)$.

   (a) Find all the fixed points (i.e. constant solutions).
   The fixed points satisfy
$$0 = x'(t) = (1+2t)(x^2-1) = (1+2t)(x-1)(x+1),$$
   so we must ahve $x = \pm 1$.

   (b) Find the general solution. (Hint: separate variables.)
   Separating the variable and integrating, we find
$$\begin{aligned} t^2+t+c &= \int (1+2t)dt = \int \frac{x'dt}{x^2-1} = \int \frac{dx}{x^2-1} \\ &= \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1) = \log(\sqrt{\frac{x-1}{x+1}}). \end{aligned}$$
   We can rearrange this to read
$$\frac{x-1}{x+1} = e^{2t^2+2t+c},$$
   or
$$x = -\frac{e^{2t^2+2t+c}+1}{e^{2t^2+2t+c}-1}.$$

3. Consider the differential equation $x' = (e^x - 1)(1 - x^2)$.

   (a) Find all the fixed points (constant solutions).
   The fixed points satisfy
$$0 = x'(t) = (e^x-1)(1-x^2) = (e^x-1)(x-1)(x+1),$$
   so we must have $x = 0, \pm 1$.

   (b) Sketch the phase portrait of this differential equation.

   (c) Classify which of these fixed points is stable.
   We have that $x' = 0$ at $x = 0, \pm 1$, and otherwise x' must have a definite sign. We can find this sign by evaluating the right hand side $(e^x-1)(1-x^2)$ at points inbetween the critical points. Then we find that $x' > 0$ for $x < -1$, $x' < 0$ for $-1 < x < 0$, $x' < 0$ for $0 < x < 1$, and $x' < 0$ for $x > 1$. Putting this all together, we can see that $x = \pm 1$ are both stable (strictly stable, in fact) and $x = 0$ is unstable. We can also see this from the phase portrait.

4. Consider the differential equation $x'' - 5x' + 4x = 0$.

   (a) Verify that $x_1 = e^{4t} + e^t$, $x_2 = e^{4t} - e^t$ and $x_3 = 2e^{4t} - 2e^t$ are all solutions to the equation.
   One can check this by taking derivatives of x_1, x_2, and x_3 and plugging the results into the left hand side of the differential equation. Upon doing this, one will find 0.

   (b) Do $\{x_1, x_2\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)
   Yes. Recall that the Wronskian satisfies a linear, homogeneous DE, so it is either always zero or never zero. Evaluating at $t = 0$, we have

$$W(0) = x_1(0)x_2'(0) - x_2(0)x_1'(0) = 2 \cdot 3 - 0 \cdot 5 = 6 \neq 0.$$

   (c) Do $\{x_1, x_3\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)
   Yes. One way to show this is to use the previous problem and the fact that $x_3 = 2x_2$. But we'll go ahead and compute the Wronskian anyhow.

$$W(0) = x_1(0)x_3'(0) - x_3(0)x_1'(0) = 2 \cdot 6 - 0 \cdot 5 = 12 \neq 0.$$

5. Consider a general second order linear differential equation of the form $x'' + p(t)x' + q(t)x = 0$.

   (a) Prove that the space of solutions is a two-dimensional vector space.
   There are two salient facts to recall: existence and uniqueness of solutions to the initial value problem, and the super-postition principle. The existence and uniqueness says that there exists a unique solution to the initial value problem

$$x'' + px' + qx = 0, \quad x(0) = a, \quad x'(0) = b$$

   for all real numbers a and b. The super-position principle states that if x_1 and x_2 are solutions then so is any linear combination

$$c_1x_1 + c_2x_2$$

   is also a solution. The super-position principle states that the solution space is a vector space. To recover the dimension, we argue as follows. The dimension must be at least 2, because the solution to the initial value problem

$$x(0) = 1, \quad x'(0) = 0$$

   is linearly independent from the solution to the initial value problem

$$x(0) = 0, \quad x'(0) = 1.$$

   Now suppose there are three linearly independent solutions x_1, x_2, x_3. Let

$$a_j = x_j(0), \qquad b_j = x_j'(0).$$

   Then one can write any solution to the initial value problem as a unique linear combination

$$x = c_1x_1 + c_2x_2 + c_3x_3.$$

   This means there is a unique solution to linear system

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0,$$

   which is impossible for a 2×3 system.

   (b) Prove that if x_1 and x_2 are solutions and they are linearly independent at $t = 0$ (i.e. one can write the solution to any initial value problem with $x(0) = a$ and $x'(0) = b$ as a linear combination of x_1 and x_2) then x_1 and x_2 are linearly independent for all t.
   Suppose two solutions x_1 and x_2 are linearly independent at $t = 0$. This means the Wronskian is nonzero at $t = 0$, or

$$W(0) = x_1(0)x_2'(0) - x_2(0)x_1'(0) \neq 0.$$

   However, the Wronskian satisfies the differential equation

$$W' = pW,$$

   so it is either always zero or never zero. Since $W(0) \neq 0$, we must have $W(t) \neq 0$ for all t, which means that x_1 and x_2 are linearly independent for all t.

6. Consider the differential equation

$$x'' - 6x' + 9x = \sin t.$$

(a) Find the general solution to the associated homogeneous problem.
The homogeneous problem is

$$x'' - 6x' + 9x = 0,$$

which has a characteristic polynomial

$$0 = r^2 - 6r + 9 = (r-3)^2.$$

Thus the general solution is

$$x(t) = c_1 e^{3t} + c_2 t e^{3t}.$$

(b) Find a solution to the inhomogeneous problem using your favorite method.
Let's try to guess a solution of the form $x = A \sin t + B \sin t$. Then we have

$$\begin{aligned} \sin t &= x'' - 6x' + 9x \\ &= -A \sin t - B \cos t - 6A \cos t + 6B \sin t + 9A \sin t + 9B \cos t \\ &= \sin t(-A + 6B + 9A) + \cos t(-B - 6A + 9B). \end{aligned}$$

. Thus we have

$$8A + 6B = 1, \qquad -6A + 8B = 0,$$

and so $A = 4/50$, $B = 3/50$. Thus a particular solution is

$$x_p = (4/50) \sin t + (3/50) \cos t.$$

(c) Find the solution to the inhomogeneous initial value problem with $x(0) = 1$ and $x'(0) = -1$.
We can write any solution in the form

$$x = c_1 x_1 + c_2 x_2 + x_p$$

where

$$x_1 = e^{3t}, \quad x_2 = te^{3t}, \quad x_p = (4/50) \sin t + (3/50) \cos t.$$

Now we need to match the coefficients to get

$$1 = x(0) = c_1 x_1(0) + c_2 x_2(0) + x_p(0) = c_1 + 3/50$$

and

$$-1 = x'(0) = c_1 x_1'(0) + c_2 x_2'(0) + x_p'(0) = 3c_1 + c_2 + 4/50.$$

Thus we have

$$c_1 = 47/50, \quad c_2 = -195/50 = -39/10,$$

and

$$x = (47/50)e^{3t} - (39/10)te^{3t} + (4/50) \sin t + (3/50) \cos t.$$

7. Consider the differential equation

$$x'' - 2x' + x = \frac{1}{1+e^t}.$$

(a) Find the general solution to the associated homogeneous problem.
The associated homogeneous problem is

$$x'' - 2x' + x = 0,$$

which has the characteristic polynomial

$$0 = r^2 - 2r + 1 = (r-1)^2.$$

Thus the general solution is

$$x = c_1 x_1 + c_2 x_2 = c_1 e^t + c_2 t e^t.$$

(b) Find a solution to the inhomogeneous problem using your favorite method.
This time we cannot use the method of undetermined coefficients (why?) and must use variation of parameters. We will look for a solution x_p of the form $x_p(t) = c_1(t)x_1(t) + c_2(t)x_2(t)$. Moreover, we will assume $c_1'x_1 + c_2'x_2 = 0$. Then $x_p' = c_1x_1' + c_2x_2' = c_1e^t + c_2(t+1)e^t$ and $x_p'' = c_1'x_1' + c_2'x_2' + c_1(2x_1' - x_1) + c_2(2x_2' - x_2) = (c_1' + c_1)e^t + c_2'(t+1)e^t + c_2(t+2)e^t$. Plugging this into the equation we find that

$$x_p'' - 2x_p' + x_p = c_1'e^t + c_2'(t+1)e^t = \frac{1}{1+e^t}.$$

We combine this equation with $c_1'e^t + c_2'te^t$ and integrate to get

$$x_p = -e^t \int e^{-2t} \cdot te^t \frac{1}{1+e^t} dt + te^t \int e^{-2t} \cdot e^t \frac{1}{1+e^t} dt = -e^t \int \frac{tdt}{e^t + e^{2t}} + te^t \int \frac{dt}{e^t + e^{2t}}.$$

(c) Find the solution to the inhomogeneous initial value problem with $x(0) = 2$ and $x'(0) = 0$.
First notice that $x_p(0) = 0 = x_p'(0)$. We can write any solution as

$$x = c_1x_2 + c_2x_2 + x_p,$$

so we only need to match the coefficients c_1 and c_2 with the initial conditions. We have

$$2 = x(0) = c_1, \quad 0 = x'(0) = c_1 + c_2,$$

so $c_1 = 2, c_2 = -2$. Thus our solution is

$$x = 2x_1 - 2x_2 + x_p.$$

8. Consider the system of differential equations

$$x' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} x.$$

(a) Find the general solution using your favorite method.
The coefficient matrix is

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = 1, \lambda_2 = 4$. THe eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(associated to $\lambda_1 = 1$) and

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(associated to $\lambda_2 = 4$). Thus the general solution is

$$x = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2 = c_1e^t \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Find the solution to the initial value problem with $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
We have to find c_1, c_2 in the expression for the general solution above to match the initial condition. However, the initial condition *is* the eigenvector v_2, so our solution is just

$$x = e^{\lambda_2 t}v_2 = e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c) What is the stability behavior of the fixed point $x = 0$? Explain your answer.
Both the eigenvalues of the coefficient matrix are positive, so the origin $x = 0$ is unstable.

9. Consider the one-parameter family of differential equations with parameter α given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_1 - \alpha x_1x_2 \\ x_2 + \alpha x_1x_2 \end{pmatrix} = F(x_1, x_2).$$

(a) Verify that the only fixed points of this system are $(0,0)$ and $(-1/\alpha, 1/\alpha)$ for $\alpha \neq 0$, and only $(0,0)$ for $\alpha = 0$.
Fixed points occur when $F(x_1, x_2) = 0$, which is equivalent to

$$0 = x_1(1 - \alpha x_2), \qquad 0 = x_2(1 + \alpha x_1).$$

This occurs precisely at

$$(0,0), \qquad (-1/\alpha, 1/\alpha).$$

(b) Linearize this system about each fixed point.
The coefficient matrix of the linearization is given by the derivative of F, which is

$$DF = \begin{bmatrix} 1 - \alpha x_2 & -\alpha x_1 \\ \alpha x_2 & 1 + \alpha x_1 \end{bmatrix}.$$

Evaulating this at $(0,0)$, we have

$$DF(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Evaulating this at $(-1/\alpha, 1/\alpha)$, we have

$$DF(-1/\alpha, 1/\alpha) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(c) What is the stability behavior of each fixed point? Explain your answer.
The eigenvalues of $DF(0,0)$ are both 1, so this fixed point is unstable. The eigenvalues of $DF(-1/\alpha, 1/\alpha)$ are ± 1, so this fixed point is also unstable (one positive eigenvalue).

(d) Draw some representative phase portraits.

(e) Is there a bifurcation point in α? If there is, find it. Explain your answer.
Yes; there is only one fixed point for $\alpha = 0$, while there are two fixed points for $\alpha \neq 0$.

10. Consider the one-parameter family of differential equations with parameter α given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} (e^{x_1} - 1)(\alpha - x_2) \\ (e^{x_2} - 1)(\alpha - x_1) \end{pmatrix} = F(x_1, x_2).$$

(a) Verify that the only fixed points of this system are $(0,0)$ and (α, α).
The fized points occur when

$$0 = (e^{x_1} - 1)(\alpha - x_2), \qquad 0 = (e^{x_2} - 1)(\alpha - x_1),$$

and this happens precisely when $(x_1, x_2) = (0,0)$ or $(x_1, x_2) = (\alpha, \alpha)$.

(b) Linearize this system about each fixed point.
The coefficient matrix of the linearization is given by

$$DF = \begin{bmatrix} (\alpha - x_2)e_1^x & 1 - e^{x_1} \\ 1 - e^{x_2} & (\alpha - x_1)e^{x_2} \end{bmatrix}.$$

Evaluating at $(0,0)$, we have

$$DF(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix},$$

Evaluating at (α, α), we have

$$DF(\alpha, \alpha) = \begin{bmatrix} 0 & 1 - e^\alpha \\ 1 - e^\alpha & 0 \end{bmatrix}.$$

(c) What is the stability behavior of each fixed point? Explain your answer.
The eigenvalues of $DF(\alpha, \alpha)$ are $\pm(1 - e^\alpha)$, which assumes both signs. Thus (α, α) is always unstable. The eigenvalues of $DF(0,0)$ are both α, which means that the system is stable if and only if $\alpha < 0$.

(d) Draw some representative phase portraits.

(e) Is there a bifurcation point in α? If there is, find it. Explain your answer.
Yes, the stability behavior of the fixed point $(0,0)$ changes as α passes from positive to negative.

11. Let $f(x) = e^{x^2}$ for $-\pi \le x \le \pi$

   (a) Is f an even function, an odd function, or neither?
   Note that
   $$f(-x) = e^{(-x)^2} = e^{x^2} = f(x),$$
   so f is even.

   (b) **Without computing them,** what can you say about the Fourier coefficients b_k in the Fourier series for f?
   We know that
   $$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx.$$
   The integrand is the product of an even function and an odd function, so it is odd. Thus the integral is zero, so all the b_k's are zero.

12. Let f be an odd function on $[-\pi, \pi]$.

   (a) What is the average value of f?
   The average value is
   $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0.$$

   (b) What can you say about the Fourier coefficients of f?
   We know that $a_0 = 0$, because this is the average value. Also,
   $$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx,$$
   which is the integral of an odd function. Thus $a_k = 0$ for all k.

   (c) Suppose that in addition f is π-periodic. Can you say anything more about the Fourier series of f?
   If f is π-periodic, then in fact we also have $b_1 = 0$.

13. For which of the PDEs listed below does the principle of superposition hold (i.e. when can you add solutions to obtain another solution)?

   (a) $u_{tt} = u_{xt}$ yes, linear and homogeneous
   (b) $u_{tt} = u_x - u_t$ yes, linear and homogeneous
   (c) $u_t^2 = u_{xx}$ no, nonlinear
   (d) $u_{tx} = u_{xx} - u_{tt}$ yes, linear and homogeneous

14. Let f and g be the 2π-periodic functions defined in the interval $[-\pi, \pi]$ by
$$f(t) = \begin{cases} \pi + t & -\pi \le t \le 0 \\ \pi - t & 0 \le t \le \pi \end{cases}$$
and
$$g(t) = \begin{cases} 1 & -\pi \le t \le 0 \\ -1 & 0 \le t \le \pi. \end{cases}$$

   (a) Find the Fourier series for f.
   First notice that f is even, so $b_n = 0$ for all n. We have
   $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_0^{\pi}(\pi - t)dt = \frac{1}{\pi}\left[\pi t - t^2/2\right]\Big|_0^{\pi} = \frac{\pi}{2}.$$
   Also, for $n \ge 1$ we have
   $$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos(nt)dt = -\frac{2}{\pi}\int_0^{\pi} t\cos(nt)dt \\ &= -\frac{2}{\pi}[\frac{t}{n}\sin(nt)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nt)dt] = \frac{2}{n\pi}\int_0^{\pi}\sin(nt)dt \\ &= \frac{2}{n^2\pi}\cos(nt)|_0^{\pi} = \frac{2((-1)^n - 1)}{n^2\pi}. \end{aligned}$$

This last quantity is $-4/(n^2\pi)$ for n odd and 0 for n even. Thus

$$f(t) = \frac{\pi}{2} - \sum_{n \text{ odd}} \frac{4}{n^2\pi}\cos(nt).$$

(b) Find the Fourier series for g.
Observe that this time g is odd, so $a_n = 0$ for all n. We compute b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)dt = -\frac{2}{\pi}\int_0^{\pi}\sin(nt)dt = \frac{2}{n\pi}\cos(nt)|_0^{\pi} \\ &= \frac{2(1-(-1)^n)}{n\pi}. \end{aligned}$$

This last quantity is $4/(n\pi)$ for n odd and 0 for n even, so

$$g = \sum_{n \text{ odd}} \frac{4}{n\pi}\sin(nt).$$

(c) Compute the derivative of f formally by differentiating its Fourier series term by term. You should obtain the Fourier series for g. Use this series expansion to verify that f is piecewise C^1 (i.e. has one piecewise continuous derivative). The key point is that the formal derivative you compute by differentiating the Fourier series term by term does actually converge.
Each term in the series for f is

$$-\frac{4}{n^2\pi}\cos(nt),$$

so the taking the derivative of the series for f term by term, we obtain

$$\sum_{n \text{ odd}} -\frac{4}{n^2\pi}(-n\sin(nt)) = \sum_{n \text{ oddd}} \frac{4}{n\pi}\sin(nt) = g(t).$$

(d) Formally differentiate f once more by differentiating the Fourier series (again) term by term. Does the series you obtain converge? Explain your answer.
If we differentiate the series term by term once more, we obtain

$$\sum_{n \text{ even}} \frac{4}{\pi}\cos(nt),$$

which does not converge. In fact, if we evaluate that series at $t = 0$, we have

$$\frac{4}{\pi}(1+1+1+1+\cdots)$$

15. (a) Find the Fourier series expansion for the odd extension of $f(t)$ where $f(t) = \pi t - t^2$ for $0 \le t \le \pi$.
The coefficients in the odd Fourier series for f are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)dt.$$

We will evaluate this integral in pieces. First we have

$$2\int_0^{\pi} t\sin(nt)dt = 2[-\frac{t}{n}\cos(nt)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nt)] = \frac{2\pi(-1)^{n+1}}{n}.$$

The rest of the coeficient is

$$\begin{aligned} -\frac{2}{\pi}\int_0^{\pi} t^2\sin(nt)dt &= -\frac{2}{\pi}[-\frac{t^2}{n}\cos(nt)\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} t\cos(nt)dt] \\ &= \frac{2\pi(-1)^n}{n} - \frac{4}{n\pi}\int_0^{\pi} t\cos(nt)dt = \frac{2\pi(-1)^n}{n} - \frac{4}{n\pi}[\frac{t}{n}\sin(nt)\Big|_0^{\pi} - \frac{1}{n}\sin(nt)dt] \\ &= \frac{2\pi(-1)^n}{n} + \frac{4}{n^2\pi}\int_0^{\pi}\sin(nt)dt = \frac{2\pi(-1)^n}{n} + \frac{4(1-(-1)^n)}{n^3\pi}. \end{aligned}$$

Putting this all together, we have

$$f(t) = \sum_{n \text{ odd}} \frac{8\sin(nt)}{n^3\pi}.$$

(b) Find the Fourier series expansion for the even extension of $f(t)$ where $f(t) = \pi t - t^2$ for $0 \leq t \leq \pi$. The coefficients in the even Fourier series for f are given by

$$a_n = \frac{2}{\pi} \int_0^\pi (\pi t - t^2) \cos(nt) dt.$$

We will evaluate this integral in pieces. First we have

$$2 \int_0^\pi t \cos(nt) dt = 2[\frac{t}{n} \sin(nt)\Big|_0^\pi - \frac{1}{n} \int_0^\pi \sin(nt) dt] = \frac{2((-1)^n - 1)}{n^2}.$$

The rest of the coefficient is

$$\begin{aligned} -\frac{2}{\pi} \int_0^\pi t^2 \cos(nt) dt &= -\frac{2}{\pi}[\frac{t^2}{n} \sin(nt)\Big|_0^\pi - \frac{2}{n} \int_0^\pi t \sin(nt) dt] = \frac{4}{n\pi} \int_0^\pi t \sin(nt) dt \\ &= \frac{4}{n\pi}[-\frac{t}{n} \cos(nt)\Big|_0^\pi + \frac{1}{n} \int_0^\pi \cos(nt) dt] = \frac{4(-1)^{n+1}}{n^2}. \end{aligned}$$

We alos have to compute the average value:

$$a_0 = \frac{1}{\pi} \int_0^\pi (\pi t - t^2) dt = \frac{\pi^2}{6}$$

Putting this all together, we have

$$f(t) = \frac{\pi^2}{6} - 4 \sum_1^\infty \frac{\cos(nt)}{n^2}.$$

(c) Find the solution to the boundary value problem

$$u_t = u_{xx}$$

for $0 \leq x \leq \pi$ and $t \geq 0$ with the initial conditions $u(x, 0) = \pi x - x^2$ and boundar conditions $u(0, t) = 0 = u(\pi, t)$. (Hint: separate variables and expand in Fourier series.)
After separating variables, we see that the solution is given by

$$u(x, t) = \sum a_n e^{-n^2 t} \sin(nx),$$

where

$$\sum a_n \sin(nx)$$

is the odd Fourier series of the initial data. Thus we have

$$u(x, t) = \sum_{n \text{ odd}} \frac{8}{n^3 \pi} e^{-n^2 t} \sin(nx).$$

(d) Find the solution to the boundary value problem

$$u_t = u_{xx}$$

for $0 \leq x \leq \pi$ and $t \geq 0$ with the initial conditions $u(x, 0) = \pi x - x^2$ and coundary conditions $u_x(0, t) = 0 = u_x(\pi, t)$. (Hint: separate variables and expand in Fourier series.)
After separating variables, we see that the solution has the form

$$u(x, t) = a_0 + \sum_1^\infty a_n e^{-n^2 t} \cos(nx),$$

where

$$a_0 + \sum a_n \cos(nx)$$

is the even Fourier series of the inital data. We have just computed that series, so we know

$$u(x, t) = \frac{\pi^2}{6} - 4 \sum_1^\infty \frac{\cos(nt)}{n^2}.$$

16. (a) Find the Fourier series expansion for the odd extension of $f(t)$ where

$$f(t) = \begin{cases} t & 0 \leq t \leq \frac{\pi}{2} \\ \pi - t & \frac{\pi}{2} \leq t \leq \pi \end{cases}$$

for $0 \leq t \leq \pi$.

The Fourier coefficients are

$$b_n = \frac{2}{\pi} \int_0^\pi f(t) \sin(nt) dt.$$

First observe that f is even under reflection through the line $x = \pi/2$, while for n even $\sin(nt)$ is odd under that reflection. Thus $b_n = 0$ for n even. Also, for n odd we have

$$b_n = \frac{4}{\pi} \int_0^{\pi/2} t \sin(nt) dt = \frac{4}{\pi} [- \frac{t}{n} \cos(nt) \Big|_0^{\pi/2} + \frac{1}{n} \int_0^{\pi/2} \cos(nt) dt] = -\frac{4}{n^2 \pi}.$$

. Thus we have

$$f(t) = -\frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n^2}.$$

(b) Solve the boundary value problem

$$u_{tt} = u_{xx}$$

for $0 \leq x \leq \pi$ and $t \geq 0$ with the initial conditions $u(x, 0) = f(x)$ and boundary conditions $u(0, t) = 0 = u(\pi, t)$. (Hint: separate variables and expand in Fourier series.)

I should have said here that the initial velocity is zero.

After separating variables, we see that we can write the solution as

$$u(x, t) = \sum a_n \sin(nx) \cos(nt),$$

where

$$\sum a_n \sin(nx)$$

is the Fourier sine series of the initial position. Thus we have

$$u(x, t) = -\frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx) \cos(nt)}{n^2}.$$

(c) Write the solution you found in the previous part as the sum of a left-traveling wave and a right-traveling wave.

By D'Alembert's principle, we can write

$$u(x, t) = -\frac{2}{\pi} \sum \frac{\sin(nx + nt) + \sin(nx - nt)}{n^2}.$$

The first term in the sum is the left moving wave, and the second term is the right moving wave.

17. Consider the wave equation

$$\partial_t^2 u = \partial_x^2 u, \qquad u(0, t) = 0 = u(1, t),$$

where the intial velocity $\partial_t u(x, 0)$ is zero and the initial position is given by

$$u(x, 0) = \begin{cases} 0 & 0 \leq x < 1/4 \\ 1 & 1/4 \leq x \leq 3/4 \\ 0 & 3/4 < x \leq 1. \end{cases}$$

What is the smallest positive time t such that $u(7/8, t) \neq 0$? (Hint: use d'Alembert's solution.)

By D'Alembert's solution, the solution is the sum of the intial wave moving left and the initial wave moving right. Moreover, both these waves move with speed 1. The wave moving to the right is closer to the point $x = 7/8$, so it will hit $x = 7/8$ first. Moreover, $x = 7/8$ cotact the right-moving wave at its right edge, which has to move a distance of $7/8 - 3/4 = 1/8$. Therefore, the time elapsed is $1/8$.

18. Solve the following heat equations.

(a) $u_t = u_{xx}$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = 2\sin(2x) - \sin(3x)$.
Because of the boundary conditions, we can write the general solution as

$$u(x,t) = \sum_1^\infty a_n e^{-n^2 t}\sin(nx),$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi u(x,0)\sin(nx)dx.$$

However, the initial data $u(x,0)$ is already in Fourier series, so we don't have to do any of these integrals. We can read off that the nonzero Fourier coefficients are

$$a_2 = 2, \qquad a_3 = -1,$$

so

$$u(x,t) = 2e^{-4t}\sin(2x) - e^{-9t}\sin(3x).$$

(b) $u_t = u_{xx}$, $u(0,t) = 0 = u(1,t)$, $u(x,0) = 1/2 - |x - 1/2|$.
Again, we have the general form of the solution:

$$u(x,t) = \sum_1^\infty a_n e^{-n^2\pi^2 t}\sin(n\pi x),$$

where a_n are the Fourier coefficients of the initial data. This time we have to do some work to compute a_n. However, we should still think before we compute. Observe that $f(x) = 1/2 - |x - 1/2|$ is even under reflection through the line $x = 1/2$. However, the function $\sin(n\pi x)$ is odd inder this reflection for n even, which implies $a_n = 0$ for n even. For n odd we can still use the symmetry. In this case, we have

$$\begin{aligned}
a_n &= \int_0^1 (1/2 - |x - 1/2|)\sin(n\pi x)dx = 2\int_0^{1/2} x\sin(n\pi x)dx \\
&= 2[-\frac{x}{n\pi}\cos(n\pi x)\Big|_0^{1/2} + \frac{1}{n\pi}\int_0^{1/2}\cos(n\pi x)dx] = 2[-\frac{\cos((n\pi)/2)}{2n\pi} + \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^{1/2}] \\
&= \frac{2\sin((n\pi)/2)}{n^2\pi^2} = \frac{2(-1)^n}{n^2\pi^2} = -\frac{2}{n^2\pi^2}.
\end{aligned}$$

Thus we have

$$u(x,t) = \sum_{n \text{ odd}} \frac{-2}{n^2\pi^2}e^{-n^2\pi^2 t}\sin(n\pi x).$$

(c) $u_t = uxx$, $u_x(0,t) = 0 = u_x(1,t)$, $u(x,0) = -\sin(2\pi x) + 2\sin(\pi x)$.
This time, because of the boundary conditions, the general solution has the form

$$u(x,t) = a_0 + \sum_1^\infty e^{-n^2\pi^2 t}\cos(n\pi x).$$

We have the compute the Fourier coefficients of the even extention of $u(x,0) = -\sin(2\pi x) + 2\sin(\pi x)$. The average value is

$$a_0 = \int_0^1 (-\sin(2\pi x) + 2\sin(\pi x))dx = -\frac{2}{\pi}\cos(\pi x)\Big|_0^1 = \frac{4}{\pi}.$$

It's easiest to break the computation of the other coefficients into pieces. First,

$$\begin{aligned}
4\int_0^1 \sin(\pi x)\cos(n\pi x)dx &= 2\int_0^1 [\sin(\pi x + n\pi x) + \sin(\pi x - n\pi x)]dx \\
&= 2[-\frac{1}{(n+1)\pi}\cos((n+1)\pi x)|_0^1 + \frac{1}{(n-1)\pi}\cos((1-n)\pi x)|_0^1] \\
&= 2[\frac{1 - (-1)^{n+1}}{(n+1)\pi} + \frac{(-1)^{n-1} - 1}{(n-1)\pi}] \\
&= 2((-1)^{n+1} - 1)\left[\frac{1}{(n-1)\pi} - \frac{1}{(n+1)\pi}\right] = \frac{4((-1)^{n+1} - 1)}{(n^2 - 1)\pi}.
\end{aligned}$$

This last quantity is $-8/((n^2-1)\pi)$ if n is even, and 0 if n is odd. Similarly, we have

$$-2\int_0^1 \sin(2\pi x)\cos(n\pi x)dx = \frac{4(1-(-1)^{n-2})}{\pi(n^2-4)}.$$

This quantity is 0 for n even and $8/(\pi(n^2-4))$ for n odd. Thus we have

$$u(x,t) = \frac{4}{\pi} + \sum_{n \text{ odd}} \frac{8}{\pi(n^2-4)} e^{-n^2\pi^2 t}\cos(n\pi x) - \sum_{n \text{ even}} \frac{8}{\pi(n^2-1)} e^{-n^2\pi^2 t}\cos(n\pi x).$$

19. (a) Consider the heat equation

$$\partial_t u = \partial_x^2 u, \qquad u(0,t) = 0 = u(L,t), \qquad u(x,0) = f(x)$$

and define the energy at time t as

$$E(t) := \int_0^L u^2(x,t)dx.$$

Show that the energy is a decreasing function in time. Does this make sense physically? (Hint: how can you characterize decreasing functions of one variable? You might want to use the equation and the boundary conditions, and integrate by parts at some point.)

The energy is decreasing if $E'(t) < 0$. We have

$$E'(t) = \frac{d}{dt}\int_0^L u^2(x,t)dx = \int_0^L \frac{\partial u^2}{\partial t}dx = \int_0^L 2uu_t = \int_0^L 2uu_{xx} = 2uu_x\Big|_{x=0}^{x=L} - \int_0^L u_x^2 dx = -\int_0^L u_x^2 dx < 0.$$

(b) Consider the same heat equation as in the previous part, but this time with the boundary conditions

$$\partial_x u(0,t) = 0 = \partial_x(L,t).$$

Is the energy still decreasing? Explain your answer.

Yes, the energy is still decreasing, because the boundary term in the integration by parts argument above is still zero.

20. The equation of motion for a damped vibrating string is

$$\partial_t^2 u + k\partial_t u = c^2\partial_x^2 u.$$

As in the case of a free vibrating string, we will consider the case where the endpoints of the string are fixed at the same height, the length of the string is L, and look for a solution of the form

$$u(x,t) = v(x)w(t).$$

(a) What are the ODEs which v and w satisfy?

Plug $u(x,t) = v(x)w(t)$ into the equation:

$$v(w'' + kw') = c^2 v'' w.$$

As in the wave equation, we divide both sides by both v and $c^2 w$ to get

$$\frac{v''}{v}(x) = \frac{w'' + kw'}{c^2 w}(t).$$

Because the left hand side is a function of x alone and the right hand side is a function of t alone, they must both be equal to some constant we'll call $-\lambda^2$. Thus we have two ODEs:

$$v'' + \lambda^2 v = 0 \qquad w'' + kw' + \lambda^2 c^2 w = 0.$$

(b) What boundary conditions and/or initial conditions must v and w satisfy?

First look at the conditions for v: we know $u(0,t) = 0 = v(0)w(t)$. So we must have $v(0) = 0$. Similarly, $u(L,t) = 0$ forces $v(L) = 0$.

For w: we have

$$f(x) = u(x,0) = v(x)w(0).$$

Similarly,

$$g(x) = \partial_t u(x,0) = v(x)w'(0).$$

These are initial conditions for the ODE for w.

(c) Can you use these boundary/initial conditions to say anything about the solutions to the ODEs?

This is much like the wave equation, in that the boundary conditions for v rule out most values of λ. Indeed, we see from the condition $v(0) = 0 = v(L)$ that we must have $\lambda = k\pi/L$ for some integer k, and $v(x) = \sin(k\pi x/L)$.

Now that we know what λ is, it is straight-forward to find the general solution for w, and hence solve the equation for a damped, vibrating string.