Practice Problems

Math 2280

March 20, 2004

1. Find the general solution to the following linear systems using your favorite method.

   (a) $x' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x$

   (b) $x' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$

   (c) $x' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} x$

2. Find the solution to the following initial value problems for each linear system using your favorite method.

   (a) $x' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   (b) $x' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   (c) $x' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. Consider the system

$$x' = Ax = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x.$$

   (a) Show that $\lambda = 1$ is a defective eigenvalue. What are the algebraic and geometric multiplicities of $\lambda = 1$?

   (b) Find the eigenvectors and generalized eigenvectors of $\lambda = 1$.

   (c) Write the general solution in terms of eigenvalues, eigenvectors and generalized eigenvectors.

   (d) Compute the matrix exponential e^{At} and use it to write down the general solution.

4. Let A be an $n \times n$ matrix with real entries such that $A^k = 0$ for some $k = 1, 2, 3, \ldots$ (such matrices are called nilpotent).

   (a) Show that the eigenvalues λ of A are all 0. (Hint: take the determinant of something.)

   (b) Is the system $x' = Ax$ stable or unstable? (Hint: use what you know about the eigenvalues.)

5. Let A be an $n \times n$ matrix with real entries such that $A^k = \mathrm{Id}$ for some $k = 1, 2, 3, \ldots$ (such matrices are called idempotent).

   (a) Show that the eigenvalues λ of A satisfy $\lambda^k = 1$. (Hint: take the determinant of something.)

   (b) The roots of the equation $\lambda^k = 1$ are called the kth roots of unity. Show that they have the form

$$\lambda = e^{2i\pi/k}, e^{4i\pi/k}, \ldots, e^{2ki\pi/k} = 1.$$

   (Hint: you may use the fact that a polynomial of order k has precisely k roots.)

   (c) Is the system $x' = Ax$ stable or unstable? (Hint: Think about the positions of the roots of unity and what this says about the signs of their real parts. You may also want to use the fact that the eigenvalues occur in conjugate pairs.)

6. Let A be an $n \times n$ matrix with real entries. Also, let V_- be the span of the eigenvectors with eigenvalues having negative real part.

   (a) Show that if $x' = Ax$ and $x(0) \in V_-$ then

$$\lim_{t \to \infty} |x(t)| = 0.$$

   (b) Conclude that the restriction of the system $x' = Ax$ to V_- is strictly stable about the fixed point $x = 0$.

7. Find all fixed points and say what you can about their stability properties for each system of differential equations below. Also, sketch their phase portraits.

(a) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(b) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(c) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(d) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(e) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(f) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 - x_2^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(g) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} x_2 & -1 \\ 1 & x_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(h) $\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}' = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_1 x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

8. Consider the system $x' = \begin{bmatrix} \alpha & 1 \\ 0 & -1 \end{bmatrix} x + |x|^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\alpha \in [-1, 1]$.

(a) Find all fixed points of this system, depending on α.

(b) Characterize the stability behavior of the fixed points as it depends on α. Sketch some representative phase portraits.

(c) Is there a bifurcation point in α? If there is, find it. Explain your answer.