Solutions to the Practice Problems

Math 2280

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1. Find the general solution to the following linear systems using your favorite method.

(a) $x' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x$

First we find the eigenvalues of the matrix.

$$0 = \det(\lambda \mathrm{Id} - \mathrm{A}) = \det \begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{bmatrix} = \lambda^2 - 5\lambda - 2.$$

So the eigenvalues are

$$\lambda = \frac{5 \pm \sqrt{32}}{2}.$$

The eigenvectors are

$$v_1 = \begin{bmatrix} 4 \\ 3 + \sqrt{32} \end{bmatrix}$$

(associated to $(5 + \sqrt{32})/2$) and

$$v_2 = \begin{bmatrix} 4 \\ 3 - \sqrt{32} \end{bmatrix}$$

(associated to $(5 - \sqrt{32})/2$). Thus the general solution is

$$x(t) = e^{5t}[c_1 e^{t\sqrt{32}} v_1 + c_2 e^{-t\sqrt{32}} v_2].$$

(b) $x' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$

First we find the eigenvalues of the matrix.

$$0 = \det(\lambda \mathrm{Id} - \mathrm{A}) = \det \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1.$$

So the eigenvalues are $\lambda = \pm i$. Next we find the eigenvectors.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \pm i \begin{bmatrix} a \\ b \end{bmatrix}.$$

So the eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix},$$

which are associated to the eigenvalues i and $-i$, respectively. Therefore, the general solution is

$$x(t) = c_1 e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} (c_1 + c_2)\cos t + (c_1 - c_2) i \sin t \\ (-c_1 + c_2) i \cos t + (c_1 + c_2) \sin t \end{bmatrix}.$$

(c) $x' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} x$

We can write the matrix A as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = P + N.$$

Notice that these two matrices P and N commute (i.e. $PN = NP$), so

$$e^{tA} = e^{t(P+N)} = e^{tP} \cdot e^{tN}.$$

We can compute these exponentials easily. P is diagonal, so

$$e^{tP} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

Also, N is nilpotent ($N^2 = 0$), so

$$e^{tN} = \mathrm{Id} + \mathrm{tN} = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$e^{tA} = e^{tP}e^{tN} = \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}.$$

Finally, if x_0 is the initial condition $x(0)$, then

$$x(t) = e^{tA}x_0 = \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix} x_0.$$

2. Find the solution to the following initial value problems for each linear system using your favorite method.

(a) $x' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We first find the eigenvalues:

$$0 = \det(\lambda \mathrm{Id} - \mathrm{A}) = \det \begin{bmatrix} \lambda - 1 & -3 \\ -2 & \lambda - 4 \end{bmatrix} = \lambda^2 - 5\lambda - 2.$$

So the eigenvalues are

$$\lambda = \frac{5 \pm \sqrt{32}}{2}.$$

Next we find the eigenvectors:

$$v_1 = \begin{bmatrix} 6 \\ 3 + \sqrt{32} \end{bmatrix}$$

(associated to $(5 + \sqrt{32})/2$) and

$$v_2 = \begin{bmatrix} 6 \\ 3 - \sqrt{32} \end{bmatrix}$$

(associated to $(5 - \sqrt{32})/2$). Thus the general solution is

$$x(t) = e^{5t}[c_1 e^{t\sqrt{32}} v_1 + c_2 e^{-t\sqrt{32}} v_2].$$

Now we have to match the coefficients by looking at $x(0)$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 v_1 + c_2 v_2,$$

so

$$1 = 6c_1 + 6c_2, \qquad 0 = (3 + \sqrt{32})c_1 + (3 - \sqrt{32})c_2.$$

The second equation says

$$c_1 = \frac{-3 + \sqrt{32}}{3 + \sqrt{32}} c_2.$$

Plugging this into the first equation we have

$$c_2 = \frac{3 + \sqrt{32}}{12\sqrt{32}}.$$

Then

$$c_1 = \frac{23}{384 + 36\sqrt{32}}.$$

(b) $x' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

First we compute the eigenvalues:

$$0 = \det(\lambda \mathrm{Id} - \mathrm{A}) = \det \begin{bmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 1 \end{bmatrix} = \lambda^2 - 2\lambda + 2.$$

Thus the eigenvalues are $\lambda = 1 \pm i$. The eigenvector for $\lambda = 1 + i$ is

$$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

and the eigenvector for $\lambda = 1 - i$ is

$$v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Thus the general solution is

$$x(t) = e^t(c_1 e^{it} v_1 + c_2 e^{-it} v_2).$$

Matching the initial conditions, we see

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ i(c_1 - c_2) \end{bmatrix}.$$

Solving for c_1 and c_2, we find $c_1 = -i/2$ and $c_2 = i/2$.

(c) $x' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This matrix is nilpotent, so it's easy to compute the expoential. Indeed, $A^2 = 0$ so

$$e^{tA} = \mathrm{Id} + \mathrm{tA} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$x(t) = e^{tA} x(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+t \\ 1 \end{bmatrix}.$$

3. Consider the system

$$x' = Ax = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x.$$

(a) Show that $\lambda = 1$ is a defective eigenvalue. What are the algebraic and geometric multiplicities of $\lambda = 1$?

First we compute the eigenvalues of A:

$$0 = \det(\lambda \mathrm{I} - \mathrm{A}) = \det \begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2.$$

So the only eigenvalue is $\lambda = 1$ and it has algebraic multiplicity 2 (i.e. it is a double root of the characteristic polynomial). However, the matrix $\lambda \mathrm{I} - \mathrm{A}$ is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which has a 1–dimensional kernel. In fact, this kernel is the span of the vector $[10]^T$. Anyhow, this means the geometric multiplicity of $\lambda = 1$ is 1, which is less than the algebraic multiplicity. So $\lambda = 1$ is a defective eigenvalue.

(b) Find the eigenvectors and generalized eigenvectors of $\lambda = 1$.

The eigenvectors are

$$\ker(\mathrm{I} - \mathrm{A}) = \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{a} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where a is any real number.

The generalized eigenvalues lie in the kernel of $(\mathrm{I} - \mathrm{A})^2 = 0$, so one can take anything which is not parallel to $[10]^T$ as a generalized eigenvector. In particular, we will choose

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

below.

(c) Write the general solution in terms of eigenvalues, eigenvectors and generalized eigenvectors.

The general solution is

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 t e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(d) Compute the matrix exponential e^{At} and use it to write down the general solution.

4. Let A be an $n \times n$ matrix with real entries such that $A^k = 0$ for some $k = 1, 2, 3, \ldots$ (such matrices are called nilpotent).

   (a) Show that the eigenvalues λ of A are all 0. (Hint: take the determinant of something.)
   Suppose λ is an eigenvalue and $v \neq 0$ is an eigenvector associated to λ. So then

$$Av = \lambda v.$$

   However, $A^k = 0$, so

$$0 = A^k v = \lambda^k v.$$

   The only way this is possible is if $\lambda^k = 0$, so $\lambda = 0$.

   (b) Is the system $x' = Ax$ stable or unstable? (Hint: use what you know about the eigenvalues.)
   All the eigenvalues of A are zero, so the system is stable but not strictly stable.

5. Let A be an $n \times n$ matrix with real entries such that $A^k = \mathrm{Id}$ for some $k = 1, 2, 3, \ldots$ (such matrices are called idempotent).

   (a) Show that the eigenvalues λ of A satisfy $\lambda^k = 1$. (Hint: take the determinant of something.)
   Suppose λ is an eigenvalue and $v \neq 0$ is an eigenvector associated to λ. So then

$$Av = \lambda v.$$

   However, $A^k = \mathrm{Id}$, so

$$v = A^k v = \lambda^k v.$$

   The only way this is possible is if $\lambda^k = 1$.

   (b) The roots of the equation $\lambda^k = 1$ are called the kth roots of unity. Show that they have the form

$$\lambda = e^{2i\pi/k}, e^{4i\pi/k}, \ldots, e^{2ki\pi/k} = 1.$$

   (Hint: you may use the fact that a polynomial of order k has precisely k roots.)
   Let $\lambda_j = e^{2i\pi j/k}, j = 1, 2, \ldots, k$, and notice that

$$\lambda_j^k = e^{(2i\pi j/k)(k)} = e^{2i\pi j} = 1.$$

   So eqach λ_j is a root of the polynomial

$$\lambda^k - 1 = 0.$$

   However, this polynomial has precisely k roots, so $\lambda_1, \ldots, \lambda_k$ are all its roots.

   (c) Is the system $x' = Ax$ stable or unstable? (Hint: Think about the positions of the roots of unity and what this says about the signs of their real parts. You may also want to use the fact that the eigenvalues occur in conjugate pairs.)
   The system is unstable. At least one of $\lambda_j = e^{2i\pi j/k}$ or $\bar{\lambda}_j = e^{-2i\pi j/k} = e^{2i\pi(k-j)/k} = \lambda_{k-j}$ must have a positive real part.

6. Let A be an $n \times n$ matrix with real entries. Also, let V_- be the span of the eigenvectors with eigenvalues having negative real part.

   (a) Show that if $x' = Ax$ and $x(0) \in V_-$ then

$$\lim_{t \to \infty} |x(t)| = 0.$$

   Suppose that $x(0) \in V_-$, say

$$x(0) = a_1 v_1 + \ldots + a_k v_k$$

   where $v_1, \ldots, v_k$ are eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_k$. Then

$$x(t) = a_1 e^{\lambda_1 t} v_1 + \ldots + a_k e^{\lambda_k t} v_k.$$

   Now observe that the real parts of $\lambda_1, \ldots, \lambda_k$ are all negative, so the maximum of all these real parts is some negative number λ. Also, let $a = \max_{1 \leq j \leq k} |a_j|$. Then

$$|x(t)| = |a_1 e^{\lambda_1 t} v_1 + \ldots + |a_k e^{\lambda_k t} v_k| \leq a e^{\lambda t} \to 0,$$

   because $\lambda < 0$.

(b) Conclude that the restriction of the system $x' = Ax$ to V_- is strictly stable about the fixed point $x = 0$. We have just shown that if $x(0) \in V_-$, then $x(t) \to 0$ as $t \to \infty$. This is the definition of strict stability.

7. Find all fixed points and say what you can about their stability properties for each system of differential equations below. Also, sketch their phase portraits.

(a) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
The only fixed point is $x = 0$, and the eigenvalues of the linearization (which is the original system) about 0 are 5 and -3. Thus the system is unstable.

(b) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
The only fixed point is $x = 0$ and the eigenvalues are $\frac{-3 \pm i\sqrt{3}}{2}$, both of which have negative real part. Thus the system is strictly stable.

(c) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
The only fixed point is $x = 0$ and the eigenvalues are $\pm i$, both of which are pure imaginary. Thus the system is stable but not strictly stable.

(d) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
There are two fixed points: $(0, 0)$ and $(0, 1)$. You should find the first easily; you can find the second by setting $-2x_1 - x_2 = x_1 - x_2$, which implies $x_1 = 0$, then plugging that into $-2x_1 - x_2 + x_1^2 + x_2^2 = 0$. The linearization about $(0, 0)$ has the coefficient matrix

$$\begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix},$$

whose eigenvalues all have negative real part. Thus $x = 0$ is strictly stable. The linearization about $(0, 1)$ has the coefficient matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

whose eigenvalues are 1 and -3. Thus $(0, 1)$ is unstable.

(e) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
The only fixed points are $(0, 0)$ and $(-\frac{1}{2}, \frac{1}{2})$. Again you find these by monkeying around with the equations $-x_2 + x_1^2 + x_2^2 = 0$ and $x_1 + x_1^2 + x_2^2 = 0$. The linearization about $(0, 0)$ has the coefficient matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which has pure imaginary eigenvalues. However, the nonlinear system is unstable about $(0, 0)$, as one can see by writing the equation in polar coordinates and separating variables. The linearization about $(-\frac{1}{2}, \frac{1}{2})$ has the coefficient matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

which has eigenvalues ± 1. Thus the system is unstable about $(-\frac{1}{2}, \frac{1}{2})$ (the linearization has a positive eigenvalue).

(f) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 - x_2^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
The only fixed point is $(0, 0)$. The linearization about $(0, 0)$ has the coefficient matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which has pure imaginary eigenvalues However, the nonlinear system is unstable. For instance solutions increase in the x_1 direction and decrease in the x_2 direction. You can see this by doing the change of variables $v = x_1 + x_2$ and $w = x_1 - x_2$.

(g) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} x_2 & -1 \\ 1 & x_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

This has fixed points at $(0,0)$ and $(1,-1)$. The linearization about $(0,0)$ has the coefficient matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which has pure imaginary eigenvalues. However, the nonlinear system is unstable, for the same reason as in the previous problem. The linearization about $(1,-1)$ has the coefficient matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

which has eigenvalues ± 1. Thus the system is unstable.

(h) $\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}' = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_1 x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The only fixed point of this system is $(0,0)$. The linearization about $(0,0)$ has the coefficient matrix

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix},$$

which has negative eigenvalues. Thus $(0,0)$ is an strictly stable fixed point.

8. Consider the system $x' = \begin{bmatrix} \alpha & 1 \\ 0 & -1 \end{bmatrix} x + |x|^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\alpha \in [-1,1]$.

(a) Find all fixed points of this system, depending on α.

Suppose $x = (x_1, x_2)$ is a fixed point. Then we must have

$$\alpha x_1 + x_2 + |x|^2 = 0, \qquad -x_2 + |x|^2 = 0.$$

Subtracting these two equations, we have

$$\alpha x_1 + 2x_2 = 0.$$

In the case that $\alpha = 0$, this equation says $x_2 = 0$, and so we must also have $x_1 = 0$. So for $\alpha = 0$, the origin is the only fixed point.

In the case that $\alpha \neq 0$, we have $x_1 = -2x_2/\alpha$, so we have the equation

$$0 = -x_2 + |x|^2 = x_2 + \frac{4x_2^2}{\alpha} + x_2^2 = x_2[(4/\alpha + 1)x_2 - 1].$$

Thus $x_2 = 0, \alpha/(4+\alpha)$. The corresponding values of x_1 are $x_1 = 0, -2/(4+\alpha)$. Thus the fixed points are

$$x = (0,0), \qquad x = \frac{1}{4+\alpha}(-2, \alpha).$$

(b) Characterize the stability behavior of the fixed points as it depends on α. Sketch some representative phase portraits.

The linearization about $(0,0)$ has the coefficient matrix

$$\begin{bmatrix} \alpha & 1 \\ 0 & -1 \end{bmatrix},$$

which has eigenvalues

$$\lambda = \frac{\alpha + 1 \pm i(\alpha - 1)}{2}.$$

The important thing is that the real part of the eigenvalue is $\alpha + 1 > 0$, so the critical point is unstable. In fact, it is an unstable spiral.

The linearization about $(-2,\alpha)/(4+\alpha)$ has the coefficient matrix

$$\begin{bmatrix} \alpha^2 + 4\alpha - 4 & 3\alpha - 4 \\ -4\alpha & \alpha - 4 \end{bmatrix},$$

which has eigenvalues

$$\lambda = \frac{(\alpha^2 + 5\alpha - 8) \pm \sqrt{\alpha^4 - 30\alpha^3 + 57\alpha^2 - 64\alpha}}{2}.$$

The first term $\alpha^2 + 5\alpha - 8$ in the numerator is always negative for $-1 \leq \alpha \leq 1$ (just evaluate at $\alpha = \pm 1$). The part inside the square root is positive for $\alpha = -1$, negative for $\alpha = 1$, and 0 for $\alpha = 0$. Anyhow, the behavior of this critical point is either a stable spiral (for $0 < \alpha \leq 1$) or a saddle node (for $-1 \leq \alpha < 0$).

(c) Is there a bifurcation point in α? If there is, find it. Explain your answer.

Yes, $\alpha = 0$ is a bifurcation point for two reasons. First, you lose a critical point at $\alpha = 0$. Second, the critical point not at the origin changes from a (strictly) stable spiral to a saddle node.