Solutions to the Practice Problems
Math 2280
Febuary 2, 2004

1. For each of the following differential equations, decide whether the given function is a solution.

   (a) $y' = (x+1)(y^2-1)$, $y = -\frac{1+\exp(x^2+2x)}{1-\exp(x^2+2x)}$ (here $\exp(z) = e^z$)
   We have that
   $$y' = -\frac{(4x+4)e^{x^2+2x}}{(1-e^{x^2+2x})^2}, \qquad y^2 - 1 = \frac{4e^{x^2+2x}}{(1-e^{x^2+2x})^2}.$$
   Thus $y' = (x+2)(y^2-1)$ and it does satisfy the equation.

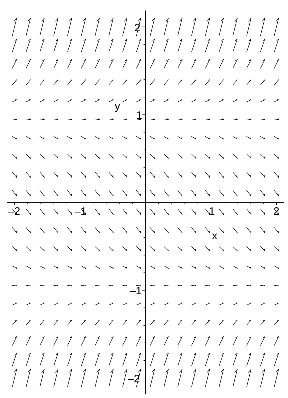
   (b) $y' + 2xy = x$, $y = 2 - e^{x^2}$
   We have that $y' = -2xe^{x^2}$, so
   $$y' + 2xy = -2xe^{x^2} + 2x(2 - e^{x^2}) = 4x(1 - e^{x^2}) \neq x.$$
   So the function does not satisfy the differential equation.

   (c) $y' = xy$, $y = e^{x^2/2}$
   We have that $y' = xe^{x^2/2} = xy$, so it does satisfy the differential equation.

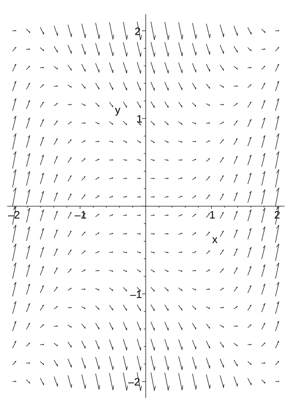
2. Sketch the slope field and some typical solution curves for each of the following differential equations.

   See me if you have questions about sketching some solution curves given the slope field. It's kind of like playing connect-the-dots.

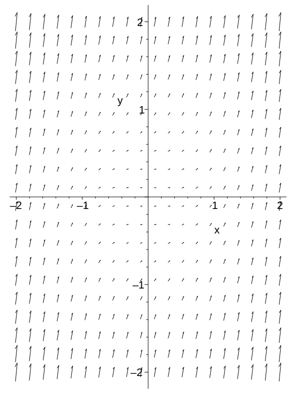
   (a) $y' = y^2 - 1$



   (b) $y' + xy = 0$ (Rotate this picture 45 degrees.)



   (c) $y' = x^2 + y^2$



3. Solve each of the given initial value problems.

   (a) $y' - 2xy = x$, $y(0) = 1$

   We multiply the equation by e^{-x^2}, and obtain
$$xe^{-x^2} = e^{-x^2}y' - 2xe^{-x^2}y = \frac{d}{dx}(ye^{-x^2}).$$
   Integrating both sides with respect to x, we have
$$e^{-x^2}y = \int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + c,$$
   and so
$$y = -\frac{1}{2} + ce^{x^2}.$$
   Plugging in the initial condition, we have
$$1 = y(0) = -\frac{1}{2} + c,$$
   and so $c = 3/2$ and
$$y = -\frac{1}{2} + \frac{3}{2}e^{x^2}.$$

   (b) $y' - x^2e^{-y} = 0$, $y(0) = 4$

   We can rearrange this equation to read
$$e^y y' = x^2,$$
   and then integrate with frespect to x. This yields
$$e^y = \frac{1}{3}x^3 + c,$$
   so
$$y = \ln(x^3/3 + c).$$
   Now we plug in $x = 0$ to find the constant c:
$$4 = y(0) = \ln(c),$$
   so $c = e^4$ and
$$y = \ln(x^3/3 + e^4).$$

   (c) $y' - ye^{yx} = 0$, $y(0) = 0$

   This one's tricky. It's an equation of the form $y' = f(x, y)$, and the initial condition happens to lie on the $f = 0$ level set. Therefore the solution is $y(x) = 0$ for all x.

   (d) $y' = y^2$, $y(0) = 1$

   We can separate variables to obtain $y'/y^2 = 1$, which we integrate to obtain
$$-\frac{1}{y} = x + c, \qquad = y = -\frac{1}{x + c}.$$
   Now plug in $x = 0$ to find the constant:
$$1 = y(0) = -\frac{1}{c}, \quad c = -1, \quad y = \frac{1}{1 - x}.$$

(e) $y' + (1/x)y = x$, $y(1) = 1$

This time we multiply by x to obtain

$$x^2 = xy' + y = \frac{d}{dx}(xy).$$

Thus

$$\frac{1}{3}x^3 + c = xy, \qquad y = \frac{1}{3}x^2 + \frac{c}{x}.$$

Plug in $x = 1$ to find the constant:

$$1 = y(1) = \frac{1}{3} + c, \quad c = \frac{2}{3}, \quad y = \frac{1}{3}x^2 + \frac{2}{3x}.$$

4. Consider the differential equation

$$y' = y(y^2 - 1).$$

(a) Find all the equilibria (constant solutions) of this equation.

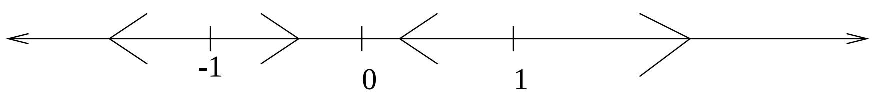
The constant solutions satisfy

$$0 = y' = y(y^2 - 1) = y(y - 1)(y + 1).$$

Therefore, we must have $y = 0, 1, -1$.

(b) Classify each of these equilibria as unstable, stable, or strictly stable.

The zeroes of $f(y) = y(y^2 - 1)$ divide the y axis into four regions: $y < -1$, $-1 < y < 0$, $0 < y < 1$ and $y > 1$. In each of these regions, $y(y^2 - 1) = y'$ can't change sign, so y is either increasing or decreasing in each region (depending on the sign of y'). For $y < -1$, $y' < 0$, and so y is decreasing in this region. For $-1 < y < 0$, $y' > 0$, so y is increasing in this region. For $0 < y < 1$, $y' < 0$, so y is decreasing in this region. For $y > 1$, $y' > 0$, so y is increasing in this region. The phase diagram for this equation looks like



Therefore $y = \pm 1$ are unstable and $y = 0$ is strictly stable.

(c) Sketch some typical solution curves.

5. (a) Write down a differential equation with one strictly stable equilibrium. (Be sure to justify your answer.)

The differential equation $y' = -y$ has one strictly stable equilibrium at $y = 0$. Indeed, $y = 0$ is the only zero fo the function on the right hand side, so it is the only equilibrium. Also, $y' > 0$ (and so y is increasing) to the left of $y = 0$. Similarly, $y' < 0$ (and so y is decreasing) to the right of $y = 0$. Thus any solution with an initial condition $y_0 \neq 0$ will tend toward 0 (exponentiall, in fact).

(b) Write down a differential equation with one stable, but not strictly stable, equilibrium. (Be sure to justify your answer.)

The differential equation

$$y' = \begin{cases} -y & y > 0 \\ 0 & y \leq 0 \end{cases}$$

has one equilibrium at $y = 0$ which is stable but not strictly stable. Indeed, solution curves which start above the horizontal axis (i.e. the equilibrium solution) get closer to the equilibrium. However, solution curves which start below the equilibrium maintain a constant distance to the equilibrium; they don't get farther away from it or closer to to.

6. Consider the differenital equation

$$y' = y(y^2 - 1) - h,$$

where h is a parameter.

(a) Does this system have a bifurcation point? If it does, find the bifucation point.

This system actually has two bifurcation points, at $h = 2/(3\sqrt{3})$ and $h = -2/(3\sqrt{3})$. One can see this by sketching the graph of the function $f(y) = y(y^2 - 1)$. The bifurcation points occur when h is the height of the local maxima/minima of the graphs of f. The function f has a local maximum at $y = -1/\sqrt{3}$, and that height is $2/(3\sqrt{3})$. Also, f has a local minimum at $y = 1/\sqrt{3}$, and that height is $-2/(3\sqrt{3})$.

(b) Describe how the behavior of the system depends on h.

We have several cases to consider.

First, when $h < -2/(3\sqrt{3})$, the right hand side has one zero, which is positive. This zero is the only equilibrium, and it is unstable. To the left of the zero, $y' < 0$ and the the right of the zero, $y' > 0$.

When $h = -2/(3\sqrt{3})$, the right hand side has two zeroes (i.e. equilibria), one of which is positive and the other negative. These are both unstable. Indeed, $y' > 0$ to the right of the positive equilibrium, $y' < 0$inbetween the two equilibria, and $y' < 0$ to the left of the negative equilibrium.

When $-2/(3\sqrt{3}) < x < 2/(3\sqrt{3})$, the function has three zeroes (i.e. equilibria). The largest and smallest of these is unstable, while the middle is strictly stable. This case is very similar (and includes) the case of problem 4b.

When $h = 2/(3\sqrt{3})$, the right hand side has two zeros (i.e. equilibria), neither of which is stable. This case is exactly like the $h = -2/(3\sqrt{3})$ case.

When $h > 2/(3\sqrt{3})$, the right hand side has one zero (i.e. equilibrium), and it is positive. This equilibrium point is unstable, because $y' > 0$ to the right of the equilibrium and $y' < 0$ to the left of the equilibrium.

(c) Sketch some typical solution curves, depending on the value of the parameter h.

7. Write down the differential equation which models the following phemonenom: the tangent line to the graph of any solution passes through the origin. Also, solve this differential equation with the initial conditions

$$y(1) = 1.$$

We want a differential equation of the form

$$y' = f(x, y).$$

Recall that $y' = f(x, y)$ is the slope of the tangent line to the graph of y, through the point (x, y). Therefore, the tangent line satisfies the equation

$$y' = \frac{y}{x},$$

because it must have slope y' and it must pass through $(0, 0)$.

This is a separable equation, and the general solution is of the form

$$\ln y = \ln x + c,$$

so

$$y = cx$$

for some constant c. By the initial condition $y(1) = 1$, we have $c = 1$ and so $y = x$.

8. Consider a falling body, where the force of wind resistance is proportial to the velocity, say

$$F_R = -m\rho v,$$

where m is the mass of the body, v its velocity (in the vertical direction) and $\rho > 0$ is a constant.

(a) Write down a differential equation for v. You may assume that the only two forces acting on the body are gravity (which is a constant force of $-mg$) and wind resistance.

The two forces are gravity (a constant force of $-mg$) and the force of wind resistance ($F_R = -m\rho v$). Therefore the acceleration is given by

$$v' = -g - \rho v.$$

(b) Find the constant solutions (i.e. equilibria) and classify them as unstable, stable, or strictly stable.

The equilibria occur when the acceleration is zero, or

$$0 = v' = -g - \rho v,$$

which implies

$$v = -\frac{g}{\rho}.$$

This is a strictly stable equilibrium. Indeed, by the equation $v' < 0$ (and hence v is decreasing) for $v > -g/\rho$ and $v' > 0$ (and hence v is increasing) for $v < -g/\rho$. Thus the solution will always approach the equilibrium point $v = -g/\rho$.

(c) Sketch the slope field and some typical solution curves.

(d) Explain your picture physically. In other words, what happens to this falling body?

Physically, the wind resistance pushes against the direction of motion. For velocities $v > -g/\rho$, the widn resistance is less than gravity, so the object approaches $v = -g/\rho$. For $v < -g/\rho$, the wind resistance is greater than gravity, so it slows the object down. In either case, the velocity approaches the terminal velocity of $v^* = -g/\rho$.

9. For each of the following differential equations, say whether the superposition principle holds. (Be sure to justify your answer.)

(a) $y'' + xy = x$ no, nonhomogeneous

(b) $y'' + x^2y' - xy = 0$ yes, linear and homogeneous

(c) $y'' - y^2 = 0$ no, nonlinear

(d) $y'' + \sin(x)y = \cos(s)y'$ yes, linear and homogeneous

10. Consider a differential equation of the form

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0.$$

(a) How many linearly independent solutions does this equation have? Why?

This equation will have three linearly independent solutions. One way to see this is the realize that one needs to specify three initial conditions for an initial value problem $(y(0), y'(0), y''(0))$, and that the solution to any such initial value problem is unique. Thus one has to specify three parameters in order to uniquely determine a solution to the differential equation.

(b) Given three solutions of this equation, how can you tell that they are linearly independent? Why does this work?

Call the solutions y_1, y_2, y_3. If these functions are linearly independent, then the only way to satisfy the equation

$$0 = c_1y_1 + c_2y_2 + c_3y_3$$

is to have $c_1 = c_2 = c_3 = 0$. Taking derivatives of the above equation, we have

$$0 = c_1y_1' + c_2y_2' + c_3y_3' = c_1y_1'' + c_2y_2'' + c_3y_3''.$$

We can write these three equation in matrix form as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

which only has the solution $c_1 = c_2 = c_3 = 0$ precisely when the determinant of the matrix is zero. In fact, one only needs to evaluate this determinant at one point (say, $x = 0$). By uniqueness of solutions to an initial value problem, the determinant is either never zero or always zero.

11. Consider a differential equation of the form

$$y'' + p(x)y' + q(x)y = 0,$$

and let y_1, y_2 be two solutions of this equation. Recall that the Wronskian is

$$W := y_1y_2' - y_2y_1'.$$

(a) The Wronskian W satisfies a differential equation. What is it?

We compute:

$$\begin{aligned} W' &= \frac{d}{dx}(y_1y_2' - y_2y_1') = y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' \\ &= y_1y_2'' - y_2y_1'' = y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) = -p(y_1y_2' - y_2y_1') \\ &= -pW. \end{aligned}$$

(b) Conclude from this differential equation that $W = W(x)$ is either never zero or always zero.

We just showed that W satisfies the equation

$$W' = -pW,$$

which has solutions

$$W(x) = W(0)\exp(\int p(x)dx).$$

The exponential part is never zero, so if $W(x) = 0$ for some x then $W(0) = 0$, and in this case W is identically zero.

12. Solve the following initial value problems.

(a) $y'' - 3y' + 4y = 0$, $y(0) = 1, y'(0) = 0$
We look for solutions of the form $y = e^{rx}$. Then

$$0 = y'' - 3y' + 4y = e^{rx}(r^2 - 3r + 4) = e^{rx}(r - (1/2)(3 + i\sqrt{7}))(r - (1/2)(3 - i\sqrt{7})).$$

Thus

$$r = \frac{3 \pm i\sqrt{7}}{2},$$

and any solution is a linear combination of

$$y_1 = \exp((x/2)(3 + i\sqrt{7})), \qquad y_2 = \exp((x/2)(3 - i\sqrt{7})).$$

Now we write this solution as $y = c_1 y_1 + c_2 y_2$ and match the initial conditions:

$$\begin{aligned} 1 &= c_1 y_1(0) + c_2 y_2(0) = c_1 + c_2 \\ 0 &= c_1 y_1'(0) + c_2 y_2'(0) = \frac{c_1(3 + i\sqrt{7})}{2} + \frac{c_2(3 - i\sqrt{7})}{2} = \frac{3}{2}(c_1 + c_2) + \frac{i\sqrt{7}}{2}(c_1 - c_2) \end{aligned}$$

Solving this linear system, we see

$$c_1 = \frac{\sqrt{7} + 3i}{2\sqrt{7}}, \qquad c_2 = \frac{\sqrt{7} - 3i}{2\sqrt{7}},$$

and so

$$y = \frac{\sqrt{7} + 3i}{2\sqrt{7}} \exp((x/2)(3 + i\sqrt{7})) + \frac{\sqrt{7} - 3i}{2\sqrt{7}} \exp((x/2)(3 - i\sqrt{7})).$$

(b) $y'' - y = 0$, $y(0) = -1, y'(0) = 1$
Again, we look for solutions of the form $y = e^{rx}$. Then

$$0 = y'' - y = e^{rx}(r^2 - 1) = e^{rx}(r - 1)(r + 1).$$

Thus $r = \pm 1$, and any solution is a linear combination of

$$y_1 = e^x, \qquad y_2 = e^{-x}.$$

Now we write the solution as $y = c_1 y_1 + c_2 y_2$ and match the initial conditions:

$$\begin{aligned} -1 &= y(0) = c_1 y_1(0) + c_2 y_2(0) = c_1 + c_2 \\ 1 &= y'(0) = c_1 y_1'(0) + c_2 y_2'(0) = c_1 - c_2. \end{aligned}$$

Solving this linear system, we see $c_1 = 0$ and $c_2 = -1$, and so

$$y = -e^{-x}.$$

(c) $y'' + 5y' + 6y = 0$, $y(0) = 1, y'(0) = 1$
Again, we look for solutions of the form $y = e^{rx}$. Then

$$0 = y'' - y = e^{rx}(r^2 + 5r + 6) = e^{rx}(r + 2)(r + 3).$$

Thus $r = -2, -3$ and any solution is a linear combination of

$$y_1 = e^{-2x}, \qquad y_2 = e^{-3x}.$$

We we write the solution as $y = c_1 y_1 + c_2 y_2$ and match the initial conditions:

$$\begin{aligned} 1 &= c_1 y_1(0) + c_2 y_2(0) = c_1 + c_2 \\ 1 &= c_1 y_1'(0) + c_2 y_2'(0) = -2c_1 - 3c_2 \end{aligned}$$

Solving this linear system, we see $c_1 = 4, c_2 = -3$, and so

$$y = 4e^{-2x} - 3e^{-3x}.$$