

Solutions to the Final Exam
Math 2280
May 6, 2004

1. Consider the differential equation (which you don't need to solve)

$$x' + 2tx^2 = 0.$$

- (a) (2 points) Is $x = \frac{1}{1-t^2}$ a solution to this differential equation?

We have

$$x' = \frac{2t}{(1-t^2)^2} = 2tx^2,$$

so x does not satisfy the differential equation.

- (b) (2 points) Is $x = \frac{1}{1+t^2}$ a solution to this differential equation?

We have

$$x' = \frac{-2t}{(1+t^2)^2} = -2tx^2,$$

so x does satisfy the differential equation. In fact, the general solution (which one can find by separating variables) is

$$x = \frac{1}{c + t^2},$$

where c is any constant.

2. Consider the differential equation

$$x'' - 4x' + 4x = 0.$$

- (a) (6 points) Find the general solution to this differential equation.

We try a solution of the form $x = e^{rt}$. Then we have

$$0 = x'' - 4x' + 4x = e^{rt}(r^2 - 4r + 4),$$

so we must have $r = 2$ (this is a double root) and $x_1 = e^{2t}$ as a solution. Because we have a double root to the characteristic polynomial, we also have $x_2 = te^{2t}$ as a solution. These two are linearly independent; indeed,

$$W(x_1, x_2) = x_1'(0)x_2(0) - x_2'(0)x_1(0) = -1 \neq 0.$$

Thus the general solution is

$$x(t) = c_1x_1(t) + c_2x_2(t) = c_1e^{2t} + c_2te^{2t}.$$

- (b) (4 points) What is the solution to this differential equation with the initial condition $x(0) = 1, x'(0) = -1$?

We will use the initial conditions to find the constants c_1 and c_2 . We have

$$1 = x(0) = c_1, \quad -1 = x'(0) = 2c_1 + c_2,$$

so $c_1 = 1$ and $c_2 = -3$. Thus the solution is

$$x = x_1 - 3x_2 = e^{2t} - 3te^{2t}.$$

3. Consider the differential equation

$$x' = (1 - e^x)(x - 1).$$

You do not need to solve this differential equation.

- (a) (6 points) Verify that $x = 0$ and $x = 1$ are the only fixed points (i.e. constant solutions) of this differential equation.

The fixed points satisfy

$$0 = x' = (1 - e^x)(x - 1),$$

so one of the factors $1 - e^x$ or $x - 1$ must be zero. The first possibility is that

$$1 - e^x = 0 \Leftrightarrow x = 0.$$

The other possibility is that

$$x - 1 = 0 \Leftrightarrow x = 1.$$

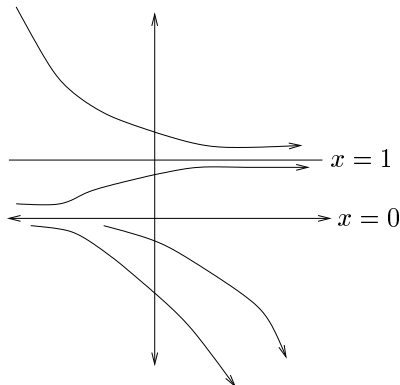
(b) (6 points) Analyze the stability properties of these fixed points.

We have found all the zeroes of x' , so x' cannot change sign on the intervals $x < 0$, $0 < x < 1$ and $x > 1$. Therefore, we only need to compute x' at a single point in these intervals to see whether x is increasing or decreasing in the entire interval. We have

$$\begin{aligned}x'(-1) &= (1 - e^{-1})(-2) < 0 \\x'(1/2) &= (1 - e^{1/2})(-1/2) > 0 \\x'(2) &= (1 - e^2)(1) < 0.\end{aligned}$$

Thus x is increasing for $0 < x < 1$ and decreasing for $x < 0$ or $x > 1$. This implies $x = 0$ is unstable and $x = 1$ is (strictly) stable.

(c) (4 points) Sketch some representative solutions to this differential equation.



4. Consider the system of differential equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -4 & -3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) (5 points) Verify that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -5$.

The eigenvalues satisfy

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -4 - \lambda & -3 \\ -1 & -2 - \lambda \end{bmatrix} = (\lambda + 4)(\lambda + 2) - 3 = \lambda^2 + 6\lambda + 5 = (\lambda + 1)(\lambda + 5).$$

Thus we have $\lambda_1 = -1$ and $\lambda_2 = -5$ as claimed.

(b) (5 points) Find the eigenvectors of A .

For the $\lambda_1 = -1$ eigenvector:

$$\begin{bmatrix} -4 & -3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} \Leftrightarrow a = -b,$$

so we can take the eigenvector to be

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For the $\lambda_2 = -5$ eigenvector:

$$\begin{bmatrix} -4 & -3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -5a \\ -5b \end{bmatrix} \Leftrightarrow a = 3b,$$

so we can take the eigenvector to be

$$v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(c) (4 points) What is the general solution to the system of differential equations $x' = Ax$?

The general solution is

$$x = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

- (d) (4 points) Notice that $x = 0$ is a fixed point (as it is for all linear, homogeneous systems). Is $x = 0$ stable or unstable? Explain your answer.

Both the eigenvalues are negative, so the system is (strictly) stable near $(0, 0)$.

5. Consider the system of differential equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_2 e^{x_1} \\ x_1 e^{x_2} \end{bmatrix} = F(x_1, x_2).$$

You do not need to solve this system.

- (a) (4 points) Verify that $(0, 0)$ is the only critical point of the system.

The critical points occur when $x_1' = 0 = x_2'$. Thus we have (because exponentials are never zero)

$$\begin{aligned} 0 &= x_1' = x_2 e^{x_1} \Leftrightarrow x_2 = 0 \\ 0 &= x_2' = x_1 e^{x_2} \Leftrightarrow x_1 = 0. \end{aligned}$$

So the only critical point is $(0, 0)$.

- (b) (4 points) What is the linearization of this system about the critical point $(0, 0)$?

The coefficient matrix of the linearization is given by

$$A = DF = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 e^{x_1} & e^{x_1} \\ e^{x_2} x_1 & e^{x_2} \end{bmatrix}.$$

Evaluating this at $(0, 0)$, we have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

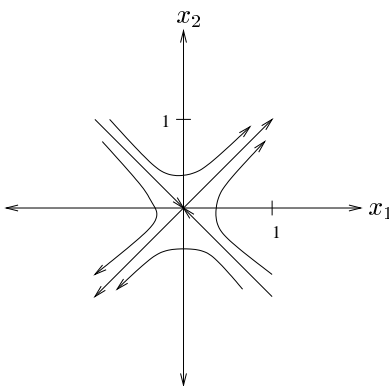
- (c) (4 points) Is the critical point $(0, 0)$ stable or unstable? Explain your answer.

The eigenvalues of the linearization satisfy

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1,$$

so the eigenvalues are $\lambda = \pm 1$. Notice that one eigenvalue is positive, so the system is unstable near $(0, 0)$. In fact, $(0, 0)$ is a saddle point.

- (d) (4 points) Sketch some representative solution curves near $(0, 0)$.



6. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is an odd, 2π -periodic function. In other words,

$$f(-x) = -f(x), \quad f(x + 2\pi) = f(x)$$

for all x .

- (a) (3 points) What is the average value of f on the interval $[-\pi, \pi]$?

The average value is

$$A = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = 0,$$

because f is an odd function.

- (b) (3 points) One can write down a Fourier series for f , by restricting f to the interval $[-\pi, \pi]$. Which coefficients in the Fourier series are zero? Explain your answer.

The a_n coefficients are zero. This is because

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx,$$

and the integrand is odd (it is the product of an even function and an odd function).

- (c) (2 points) Show that if f is differentiable, then f' is even (in other words, $f'(-x) = f'(x)$). (Hint: you might want to use the chain rule.)

There are actually two ways to argue that f' is even. The first way is to write out the Fourier series of f and differentiate term by term. The Fourier series of f is a sum of sines, so the Fourier series of f' is a sum of cosines, and hence even. This is correct reasoning, but a little clunky.

The proof below is a bit pedantic, but that's ok. We define $g(x) = -x$. Then

$$f(-x) = f(g(x)) = f \circ g(x), \quad f(x) = f(g(g(x))) = f \circ g \circ g(x).$$

Also observe that $g'(x) = -1$. So by the chain rule

$$f'(x) = \frac{d}{dx}(f \circ g \circ g)(x) = (f \circ g)'(x)g'(x) = f'(-x)(g'(x))^2 = f'(-x).$$

7. (10 points) For $0 \leq x \leq \pi$, consider the function f defined by

$$f(x) = x(\pi - x).$$

Verify that the Fourier series of the odd extension of f to $[-\pi, \pi]$ is

$$f(x) = \sum_{n \text{ odd}} \frac{8}{n^3 \pi} \sin(nx).$$

We have that

$$f(x) = \sum_1^{\infty} b_n \sin(nx),$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx = 2 \int_0^{\pi} x \sin(nx) dx - \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx) dx \\ &= 2 \left[-\frac{x}{n} \cos(nx) \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nx) dx - \frac{2}{\pi} \left[-\frac{x^2}{n} \cos(nx) \right]_0^{\pi} + \frac{2}{n} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2\pi(-1)^{n+1}}{n} + \frac{2\pi(-1)^n}{n} - \frac{4}{n\pi} \int_0^{\pi} x \cos(nx) dx = -\frac{4}{n\pi} \left[\frac{x}{n} \sin(nx) \right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin(nx) dx \\ &= \frac{4}{n^2 \pi} \int_0^{\pi} \sin(nx) dx = \frac{4}{n^3 \pi} [1 - (-1)^n]. \end{aligned}$$

This last quantity is zero when n is even, and $8/(n^3 \pi)$ when n is odd.

8. Consider the heat equation

$$u_t = u_{xx}$$

with the boundary conditions

$$u(0, t) = 0 = u(\pi, t)$$

and initial conditions

$$u(x, 0) = x(\pi - x).$$

- (a) (4 points) If one looks for a solution of the form $u(x, t) = v(x)w(t)$, then what differential equations must v and w satisfy?

We try a solution of the form $u(x, t) = v(x)w(t)$. Then the differential equation reads

$$vw' = v''w.$$

Dividing by $u = vw$, we can rearrange this as

$$\frac{w'}{w}(t) = \frac{v''}{v}(x).$$

Because the right hand side of this equation depends on x alone and the left hand side depends on t alone, both sides must be constant. Therefore, we must have some constant c so that

$$\frac{v''}{v} = c = \frac{w'}{w}.$$

We will assume the constant is negative (see part c), so we can write $c = -\lambda^2$ and

$$\frac{v''}{v} = -\lambda^2 = \frac{w'}{w}.$$

- (b) (4 points) What are the boundary conditions on the function $v(x)$?

Notice we have

$$0 = u(0, t) = v(0)w(t),$$

which implies $v(0) = 0$. Similarly,

$$0 = u(\pi, t) = v(\pi)w(t),$$

which implies $v(\pi) = 0$. So the boundary conditions on v are

$$v(0) = 0 = v(\pi).$$

- (c) (4 points) What are the general solutions to the differential equations for v and w ? You may assume that the separation constant is negative.

The ODE for v is

$$v'' = -\lambda^2 v,$$

which has solutions

$$v = a_\lambda \cos(\lambda x) + b_\lambda \sin(\lambda x).$$

The boundary conditions give us

$$0 = v(0) = a_\lambda$$

and

$$0 = v(\pi) = b_\lambda \sin(\pi\lambda) \Leftrightarrow \lambda = n = 1, 2, 3, \dots$$

Thus we have

$$-\lambda^2 = -n^2, \quad v(x) = b_n \sin(nx).$$

The ODE for w now reads

$$w' = -n^2 w,$$

which has the solution

$$w(t) = e^{-n^2 t}.$$

- (d) (3 points) What is the general solution for a heat equation of this type (i.e. these boundary conditions)? The general solution is a super-position of these sines and exponentials. So we have

$$u(x, t) = \sum_1^\infty b_n e^{-n^2 t} \sin(nx).$$

- (e) (3 points) What is the solution to the heat equation with the boundary conditions and initial conditions listed above? (Hint: you should have computed the relevant Fourier coefficients in problem 7.)

We have to find the coefficients b_n to match our initial conditions;

$$x(\pi - x) = u(x, 0) = \sum_1^\infty b_n \sin(nx).$$

Fortunately, we just computed those Fourier coefficients in problem 7, so we have

$$b_n = \frac{8}{n^3 \pi} \text{ for } n \text{ odd}, \quad b_n = 0 \text{ for } n \text{ even}.$$

Thus we have

$$u(x, t) = \sum_{n \text{ odd}} \frac{8}{n^3 \pi} e^{-n^2 t} \sin(nx).$$