

Solutions to the Second Midterm Exam  
Math 2280  
April 2, 2004

1. Consider the system of differential equations

$$x' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x = Ax.$$

(a) (6 points) What are the eigenvalues of  $A$ ?

$$0 = \det(\lambda \text{Id} - A) = \det \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

Thus the eigenvalues are  $\lambda_1 = 3$  and  $\lambda_2 = -1$ .

(b) (6 points) What are the eigenvectors of  $A$ ?

First we find the  $\lambda_1 = 3$  eigenvector.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}.$$

So we have

$$a + 2b = 3a \Rightarrow a = b,$$

so we can take this eigenvector to be

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the  $\lambda_2 = -1$  eigenvector we have

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix}.$$

So we have

$$a + 2b = -a \Rightarrow a = -b,$$

so we can take this eigenvector to be

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(c) (4 points) What is the general solution of this system of differential equations?

The general solution has the form

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

One should say something about why these two solutions we have found are linearly independent. One can evaluate the Wronskian:

$$W = \det(x_1(0)x_2(0)) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0.$$

Or one can just observe that  $A$  has distinct eigenvalues, so the eigenvectors  $v_1$  and  $v_2$  have to be linearly independent.

2. (10 points) Solve the initial value problem

$$x' = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(Hint: what is  $A^2$ ?)

Notice that  $A^2 = 0$ , so

$$e^{tA} = \text{Id} + tA = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}.$$

Thus

$$x(t) = e^{tA} x(0) = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 2t \\ 1 \end{bmatrix}.$$

3. Consider the system of differential equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} e^{x_2} - 1 \\ x_1(x_2 + 1) \end{bmatrix}.$$

(Note: you do not need to solve this system of differential equations.)

(a) (4 points) Show that  $(0, 0)$  is a critical point.

Let

$$F(x) = \begin{bmatrix} e^{x_2} - 1 \\ x_1(x_2 + 1) \end{bmatrix}.$$

Critical points occur when  $F(x) = 0$ , so we must evaluate  $F(0)$ :

$$F(0) = \begin{bmatrix} e^0 - 1 \\ 0(0 + 1) \end{bmatrix} = 0.$$

In fact, as many of you noticed,  $(0, 0)$  is the only critical point. If  $(x_1, x_2)$  is a critical point then

$$e^{x_2} - 1 = 0, \quad x_1(x_2 + 1) = 0.$$

The first equation implies  $x_2 = 0$ . This together with the second equation implies  $x_1 = 0$ .

(b) (4 points) What is the linearization of this system about  $(0, 0)$ ?

The linearization is given by

$$x' = Ax,$$

where

$$A = DF|_{(0,0)} = \begin{bmatrix} 0 & e^{x_2} \\ x_2 + 1 & x_1 \end{bmatrix} \Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus the linearization is

$$x' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x.$$

(c) (4 points) Is  $(0, 0)$  a stable or unstable critical point? Be sure to explain your answer.

We have to compute the eigenvalues of the linearized system:

$$0 = \det(\lambda \text{Id} - A) = \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1).$$

Thus there is one positive eigenvalue and one negative eigenvalue. Because there is a positive eigenvalue, the system is unstable near  $(0, 0)$ .

(d) (4 points) Sketch some representative solution curves near  $(0, 0)$ .

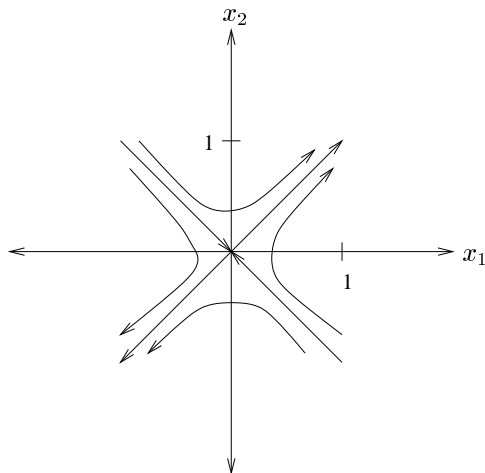
First note that eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(associated to  $\lambda = 1$ ) and

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(associated to  $\lambda = -1$ ). So the phase portrait near  $(0, 0)$  looks like



4. Consider a system of differential equations of the form

$$x' = Ax,$$

where  $A$  is an  $n \times n$  matrix. Also, let  $V_+$  be the span of the eigenvectors of  $A$  whose eigenvalues have positive real parts.

- (a) (4 points) Show that if  $x_0 \in V_+$ ,  $x_0 \neq 0$  and  $x(t)$  solves the initial value problem  $x' = Ax$ ,  $x(0) = x_0$ , then

$$\lim_{t \rightarrow \infty} |x(t)| = \infty.$$

Let  $v_1, \dots, v_k$  be a basis for  $V_+$  where each  $v_j$  is an eigenvector of  $A$  with eigenvalue  $\lambda_j$ . For convenience, we will normalize the eigenvectors so that  $|v_j| = 1$ . We can write

$$x_0 = a_1 v_1 + \dots + a_k v_k,$$

where at least one of the  $a_j$ 's are nonzero. Now let

$$a := \min_{1 \leq j \leq k} |a_j|, \quad \lambda = \min_{1 \leq j \leq k} \Re(\lambda_j).$$

We have that

$$|x(t)| = |a_1 e^{\lambda_1 t} v_1 + \dots + a_k e^{\lambda_k t} v_k| \geq a e^{\lambda t} \rightarrow \infty$$

as  $t \rightarrow \infty$ .

- (b) (4 points) If  $V_+$  is not just the zero vector, is this system stable or unstable? Be sure to explain your answer. (Hint: you may use the first part of this problem without proof if you need to.)

If  $V_+$  is not just the zero vector, we can pick  $v \in V_+$  as close as we please to 0 as our initial condition. By the first part of the problem,  $|x(t)| \rightarrow \infty$ , so the system is unstable.