Solutions to the First Midterm Exam

Math 2280

Feb. 17, 2004

1. Consider the differential equation

$$y' = y^2(x-1).$$

Note: you do not need to find the general solution to this differential equation.

(a) (2 points) Is $y = \frac{1}{1-x^2}$ a solution to this differential equation?
We will compute both sides of the differential equation:

$$\begin{aligned} y' &= \frac{d}{dx}(1-x^2)^{-1} = \frac{-1}{(1-x^2)^2}(-2x) = \frac{2x}{(1-x^2)^2} \\ y^2(x-1) &= \frac{x-1}{(1-x^2)^2} = \frac{-1}{1-x^2} \neq y'. \end{aligned}$$

Thus this choice of y does not satisfy the differential equation.

(b) (2 points) Is $y = \frac{1}{x-x^2}$ a solution to this differential equation?
Again, we will compute both sides of the equation:

$$\begin{aligned} y' &= \frac{d}{dx}(x-x^2)^{-1} = \frac{-1}{(x-x^2)^2}(1-2x) = \frac{2x-1}{(x-x^2)^2} \\ y^2(x-1) &= \frac{x-1}{(x-x^2)^2} = \frac{1-x}{x(x-x^2)} \neq y'. \end{aligned}$$

Thus this choice of y does not satisfy the differential equation either. Incidentally, one can solve this differential equation by separating variables and integrating, and verify that the general form of the solution is

$$y = \frac{1}{c+x+x^2/2},$$

where c is the constant of integration.

2. (10 points) Solve the initial value problem

$$y' + \frac{y}{x} = x^2, \qquad y(1) = 1.$$

This is a linear equation, of the form

$$y' + p(x)y = q(x).$$

If we let

$$P(x) = \int p(x)dx = \int \frac{x}{dx} = \ln x,$$

and multiply by $e^P(x)$, then the equation becomes

$$\frac{d}{dx}(xy) = xy' + y = x^3.$$

Integrating both sides with respect to x, we obtain

$$xy = \frac{x^4}{4} + c,$$

so the general form of the solution is

$$y = \frac{x^3}{4} + \frac{c}{x}.$$

Now we must mathc the initial conditions to find the constant c:

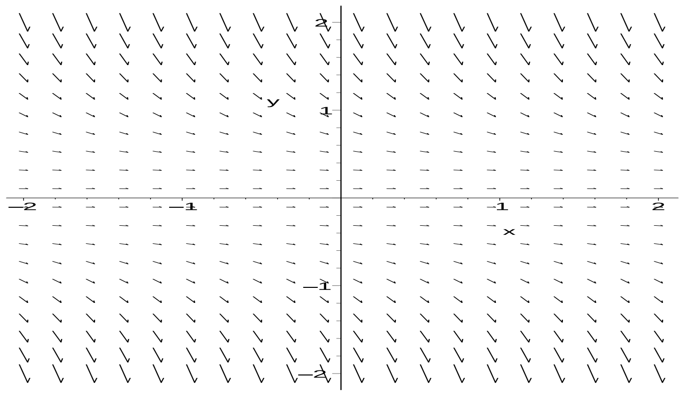
$$1 = y(1) = \frac{1}{4} + c,$$

so $c = 3/4$ and

$$y = \frac{x^3}{4} + \frac{3}{4x}.$$

3. Below I have drawn the slope field to a differential equation of the form

$$y' = f(x, y).$$



(a) (3 points) Does f depend on both x and y, or only one of them? Be sure to explain your answer.
Notice that the slope field does not depend on x. In other words, the slope remains the same as one translates horizontally. Therefroe, f is a function of y alone.

(b) (4 points) Explain why $y = 0$ is an equilibrium solution.
Notice that the slope becomes horizontal as $y \to 0$, and so we must have $f(0) = 0$. Thus any solution with $y(0) = 0$ must remain at 0 and so it is constant. Therefore, $y = 0$ is an equilibrium point.

(c) (4 points) It $y = 0$ stable?
No. The solutions with $y(0) = -\epsilon$, for any $\epsilon > 0$, have the property that

$$\lim_{x \to \infty} y(x) = -\infty.$$

Therefore, $y = 0$ is unstable.

4. Consider the differential equation

$$y' = y - y^2.$$

Note: you do not need to solve this differential equation.

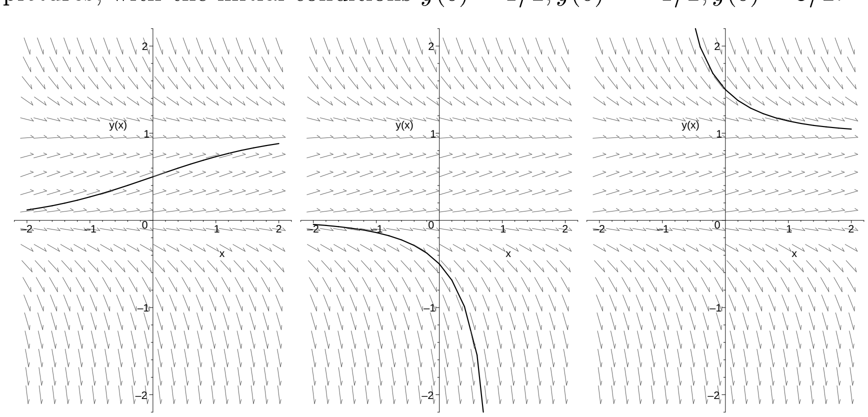
(a) (5 points) Verify that $y = 0, 1$ are the only equilibria (i.e. constant solutions) to this differential equation.
The equilibria are constant solutions, so for and equilibrium point we must have

$$0 = y' = y - y^2 = y(1 - y).$$

The only roots of this quadratic are $y = 0, 1$, so they are the only equilibria.

(b) (5 points) Classify these equilibria as unstable, stable, or strictly stable.
Let $f(y) = y - y^2$, and notice that $f(y) > 0$ if and only if $0 < y < 1$. Therefore, solutions with $0 < y(0) < 1$ are increasing, while solutions with $y(0) < 0$ or $y(0) > 1$ are decreasing. This implies that solutions with $y(0) < 0$ decrease to $-\infty$, while solutions with $0 < y(0) < 1$ or $y(0) > 1$ limit to 1. In other words, $y = 1$ is strictly stable while $y = 0$ is unstable.

(c) (5 points) Sketch some typical solution curves to this differential equation.
Here are some pictures, with the initial conditions $y(0) = 1/2, y(0) = -1/2, y(0) = 3/2$.



5. Consider the differential equation

$$y'' - 3y' + 2y = 0.$$

(a) (6 points) Find the general solution to this differential equation.
We will look for a general solution of the form $y = e^{rx}$. Then

$$0 = y'' - 3y' + 2y = r^2 e^{rx} - 3re^{rx} + 2e^{rx} = e^{rx}(r-1)(r-2).$$

Thus $r = 1, 2$ and two linearly independent solutions are

$$y_1 = e^x, \qquad y_2 = e^{2x}.$$

We can check that these functions are independent by checking the Wronskian:

$$W = y_1 y_2' - y_2 y_1' = e^{3x} \neq 0.$$

Thus, y_1 and y_2 form a basis for the solution space of the differential equation, and (by super-position), one can write the general solution as

$$y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{2x},$$

for some constants c_1 and c_2.

(b) (4 points) Find the solution to the differential equation with the initial conditions

$$y(0) = 0, \qquad y'(0) = 1.$$

We know that

$$y = c_1 e^x + c_2 e^{2x}.$$

All that remains is to match the initial conditions:

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 \\ 1 &= y'(0) = c_1 + 2c_2. \end{aligned}$$

Solving this system, we find that $c_1 = -1$ and $c_2 = 1$, so

$$y = -e^x + e^{2x}.$$