Solution to the Practice Problems

Math 2270

April 29, 2002

1. Classify the following maps as linear, affine or other.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - x$ affine

   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$ other

   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x$ linear

   (d) $f : P_2 \to P_2$, $P_2 =$ quadratic polynomials, $f(p)(t) = p'(t) - 2p(t)$. linear

   (e) $f : P_2 \to P_2$, $P_2 =$ quadratic polynomials, $f(p)(t) = p'(t) - 2p^2(t)$. other

   (f) $f : P_2 \to P_2$, $P_2 =$ quadratic polynomials, $f(p)(t) = p'(t) - t^2$. affine

2. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ which has the matrix representation (in the standard basis)

$$[T] = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}.$$

   (a) Is T invertible? Explain your answer.

   No. Notice that the sum of the first two rows is the third row, so T cannot me invertible. Alternatively, you can check that $5/4$ times the first column minus $3/4$ times the second column is the third column.

   (b) Find the kernel and range of T.

   We denote the columns of $[T]$ as v_1, v_2, v_3; by definition the range of T is the span of v_1, v_2, v_3. However, we saw in the previous part that $5/4v_1 - 3/4v_2 = v_3$. Also, v_1 and v_2 are not parallel. Putting this together we see that the range of T is the span of v_1, v_2, which is the plane through the origin with the normal vector $n = v_1 \times v_2 = (-4, -4, 4)$.

   To see what the kernel is, we reason as follows. The fact that $5/4v_1 - 3/4v_2 = v_3$ says that $(5/4, -3/4, -1) \in \ker(T)$. We already saw that the range of T is two dimensional, so by the Rank-Nullity theorem the kernel is one dimensional. Thus $\ker(T) = \text{span}\{(5/4, -3/4, -1)\}$.

   (c) For which b can you solve the equation $T(v) = b$? Is the solution ever unique?

   We can solve the equation $T(v) = b$ precisely when b is in the range of T, which is the plane through the origin with the normal vector $n = (-4, -4, 4)$. This solution is never unique because we can always add something in the kernel to our solution.

   (d) State the Rank-Nullity theorem and verify that it holds for T.

   The Rank-Nullity theorem says that the rank + the dimension of the kernel = the dimension of the domain (number of columns), which in this case is 3. We have seen that the rank is 2 and the dimension of the kernel is 1, which adds up to 3.

   (e) *Without computing any eigenvalues*, decide whether 0 is an eigenvalue of T. Explain your answer.

   Yes. 0 is an eigenvalue precisely when the kernel is nonempty. (Think about the eigenvalue-eigenvector equation.)

3. Consider the 2×2 matrix

$$T = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}.$$

   (a) For which values of θ is T invertible?

   T is invertible when $(\cos\theta, \sin\theta)$ is not parallel to $(1, 0)$, which happens precisely when $\sin\theta \neq 0$. This occurs for $\theta \neq k\pi$ for any integer k.

   (b) For which values of θ is T an orthogonal matrix?

   This happens when the columns are orthogonal, which happens when $\cos\theta = 0$ (take a dot product!). Thus we must have $\theta = \pm\pi/2, \pm 3\pi/2$, etc.

   (c) Draw a picture of the parallelogram which is the image of $[0, 1] \times [0, 1]$ under T.

   (d) For which values of θ is the area of that parallelogram 1?

   The area of the parallelogram is $|\det T| = |\sin\theta|$. This is 1 precisely when $\theta = \pm\pi/2, \pm 3\pi/2$, etc.

4. Consider the linear map $T : P_2 \to P_2$ given by

$$T(p)(t) = p(t) - p(-t),$$

where P_2 is the collection of all polynomials of degree less than or equal to two.

(a) Write down the matrix representation of T in the basis $B = \{t^2+t, t^2-t, t+1\}$.
Let $p_1 = t^2+t, p_2 = t^2-t, p_3 = t+1$. Then we see

$$T(p_1) = 2t = 2p_1 - 2p_2.$$

Similarly,

$$T(p_2) = -2t = -2p_1 + 2p_2$$

and

$$T(p_3) = 2t = 2p_1 - 2p_2.$$

Thus the matrix of T in the basis B is

$$[T]_B = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Find the kernel of T. (Hint: you do not need to use the matrix you computed in the previous part of this problem.)
Suppose $T(p) = 0$. Then $p(t) = p(-t)$ for all t; in other words p must be even. Thus we must have $p(t) = at^2 + c$ for some a, c. In other words, $\ker(T) = \text{span}\{1, \mathrm{t}^2\}$, which is two dimensional.

(c) Find the rank of T. (Hint: again, you do not need to use the matrix representation of T).
We saw that the dimension of the kernel of T is 2. By the Rank-Nullity theorem, we must have

$$3 = \text{rank(T)} + 2,$$

which implies that the rank of T is 1.

5. Consider a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$.

(a) If T is one-to-one, what can you say about the relationship between n and m?
If T is one-to-one we must have $m \leq n$.

(b) If T is onto, what can you say about the relationship between n and m?
If T is onto we must have $m \geq n$.

(c) In the particular case of $m = 4$ and $n = 2$, what are the possible pairings of the dimension of the kernel and the dimension of the range of T? For example, is it possible that the dimension of kernel of T is 3 while the dimension of the range of T is 2? List all possibilities.
The possible values of the rank are $0, 1, 2$ and the rank and nullity must sum up to 4. If the rank is 0 then the nullity must be 4; if the rank is 1 then the nullity must be 3; finally, if the rank is 2 then the nullity must be 2.

6. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

and let V be the span of $\{v_1, v_2, v_3\}$.

(a) What is the dimension of V?
Notice that $v_3 = v_1 + v_2$ and that v_1, v_2 are not parallel. Thus V is two dimensional and v_1, v_2 form a basis of V.

(b) Let T be the 4×3 matrix whose columns are v_1, v_2, v_3. What is the kernel of T?
The kernel of T is all the vectors $(a, b, c) \in \mathbb{R}^3$ such that $av_1 + bv_2 + cv_3 = 0$. Notice that $(1, 1, -1) \in \ker(T)$ (because $v_1 + v_2 = v_3$). In particular, the nullity of T is at least 1. But the sum of the rank of T and the nullity of T must be 3 (by the Rank-Nullity theorem), and the first two columns of T are not parallel, which implies $\text{rank(T)} \geq 2$. Thus we must have that the kernel of T is one dimensional, and we conclude that $\ker(T) = \text{span}\{(1, 1, -1)\}$.

(c) Apply the Gramm-Schmidt process to v_1, v_2, v_3 to obtain an orthonormal basis for V.

We only need to apply Gramm-Schmidt to v_1 and v_2 (because v_3 is a linear combination of v_1 and v_2). First we let

$$u_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

Next we let

$$\tilde{u}_2 = v_2 - \mathrm{Proj}_{\mathrm{u}_1} \mathrm{v}_2 = \mathrm{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} - \frac{1-4}{6} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \\ 1 \\ -1 \end{pmatrix}.$$

Finally, we let

$$u_2 = \frac{\tilde{u}_2}{|\tilde{u}_2|} = \frac{\sqrt{2}}{3} \begin{pmatrix} 3/2 \\ -1/2 \\ 1 \\ -1 \end{pmatrix}.$$

(d) Let

$$w = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and compute the orthogonal projection of w onto V.

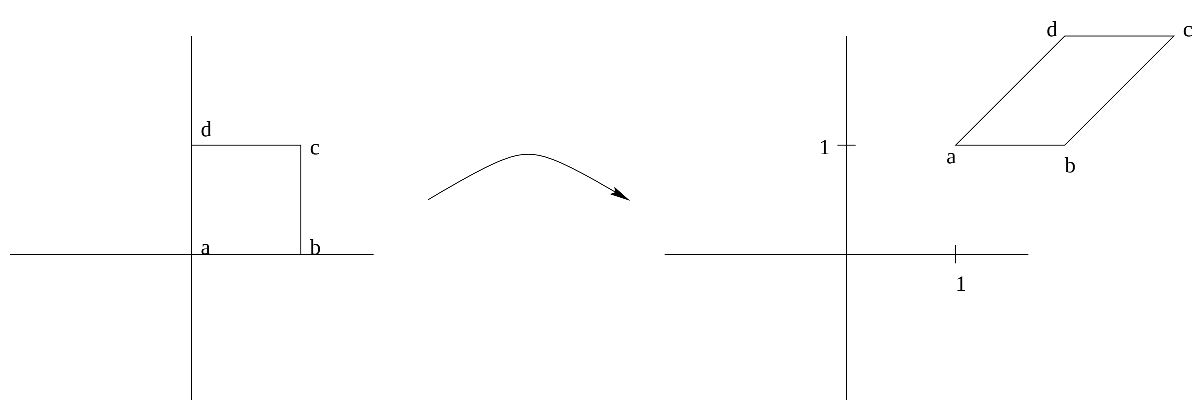
The projection is

$$\begin{aligned} \mathrm{Proj}_{\mathrm{V}} \mathrm{w} &= \mathrm{Proj}_{\mathrm{u}_1} \mathrm{w} + \mathrm{Proj}_{\mathrm{u}_2} \mathrm{w} = (\mathrm{u}_1 \cdot \mathrm{w})\mathrm{u}_1 + (\mathrm{u}_2 \cdot \mathrm{w})\mathrm{u}_2 = \frac{-1}{\sqrt{6}} \mathrm{u}_1 + \frac{1}{3\sqrt{2}} \mathrm{u}_2 \\ &= -\frac{1}{6} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{9} \begin{pmatrix} 3/2 \\ -1/2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2/9 \\ 1/9 \\ -7/18 \end{pmatrix}. \end{aligned}$$

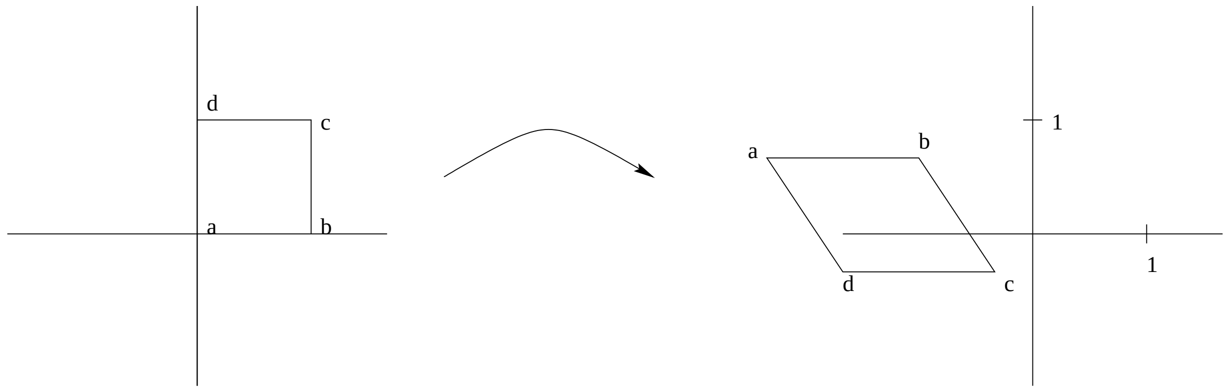
7. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map given by

$$T(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Draw the parallelogram which is the image of $[0, 1] \times [0, 1]$ under T, making sure to label the corners.



(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map which generates the following picture.



Find the matrix representation of this affine map (in the standard basis).
First note $T(0,0) = (-7/3, 2/3)$. Thus $T(v) = Av + (-7/3, 2/3)$ for some matrix A. To find A, first we translate the parallelogram so that the vertex labeled a is at the origin. Then the first column of A is the edge given by $\vec{ab} = (-1, 2/3) - (-7/3, 2/3) = (4/3, 0)$. The second column of A is given by $\vec{ad} = (-5/3, -1/3) - (-7/3, 2/3) = (2/3, -1)$. Therefore T is given by

$$T(v) = \begin{bmatrix} 4/3 & 2/3 \\ 0 & -1 \end{bmatrix} + \begin{pmatrix} -7/3 \\ 2/3 \end{pmatrix}.$$

8. Let A be a 3×3 matrix with real entries.

(a) Is it possible that $i, 1+i, 1-i$ are the eigenvalues of A?
No. Because the characteristic polynomial of A has real entries its root must occur in conjugate pairs. In particular (because there are three total roots) at least on of the roots must be real, which is not the case with this list.

(b) Is it possible that $1, 1+i, 1+2i$ are the eigenvalues of A?
No. Again, the nonreal eigenvalues (roots) must occur in conjugate pairs. But $1+i$ is not the conjugate of $1+2i$.

(c) Is it possible that $1, 1+i, 1-i$ are the eigenvalues of A?
Yes. This list has one real root and the nonreal roots are conjugate.

(d) If A is orthogonal, is it possible that $1, 1+i, 1-i$ are the eigenvalues of A?
No. The eigenvalues of an orthogonal matrix must have length 1 (see below).

(e) How many nonreal eigenvalues can A possibly have? List all possibilities.
It can have either 0 or 2 nonreal eigenvalues, because they must occur in conjugate pairs and there are at most three total eigenvalues.

9. (a) Let A be an $n \times n$ matrix with real entries such that $A^t = A$. Also, let $\lambda_1 \neq \lambda_2$ be eigenvalues of A with eigenvectors v_1 and v_2 such that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Show that $v_1 \perp v_2$.
We have that

$$\langle Av_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle.$$

At the same time,

$$\langle Av_1, v_2 \rangle = \langle v_1, A^t v_2 \rangle = \langle v_1, Av_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle.$$

Putting these together, we see

$$(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0.$$

Because $\lambda_1 - \lambda_2 \neq 0$ (by assumption) we must have $\langle v_1, v_2 \rangle = 0$, i.e. that $v_1 \perp v_2$.

(b) If A is an $n \times n$ matrix such that $A^k = 0$ for some positive integer k, show that all the eigenvalues of A must be zero. (Note: this does not mean that A is the zero matrix. Can you think of some examples of matrices A such that $A \neq 0$ but $A^2 = 0$?)
If $A^k = 0$ and v is an eigenvector of A with eigenvalue λ, then

$$0 = A^k v = \lambda^k v.$$

Because $v \neq 0$ this means we must have $\lambda = 0$.

(c) If A is an $n \times n$ matrix such that $A^k = \mathrm{Id}$ for some positive integer k, show that $|\lambda| = 1$ for all eigenvalues λ of A.
If $A^k = \mathrm{Id}$ and v is an eigenvector of A with eigenvalue λ, then

$$v = A^k v = \lambda^k v.$$

Because $v \neq 0$ this means we must have $\lambda^k = 1$, which in particular implies $|\lambda| = 1$.

(d) Let A be an $n \times n$ orthogonal matrix (i.e. $A^{-1} = A^t$). Show that $|\lambda| = 1$ for all eigenvalues λ of A.
Let v be an eigenvector of A with eigenvalue λ. Then $Av = \lambda v$ and A orthogonal imply

$$\|v\|^2 = \langle v, v\rangle = \langle Av, Av\rangle = \lambda^2 \langle v, v\rangle = \lambda^2 \|v\|^2.$$

In particular, this means $|\lambda| = 1$.

10. Consider the discrete dynamical system $x_{n+1} = Ax_n$ for some 2×2 matrix A. Notice that the origin is a fixed point of this system (i.e. the origin stays fixed). For each choice of matrix A listed below, classify the origin as unstable, stable or asymptotically stable and sketch a phase portrait for the system.

(a) $A = \begin{bmatrix} 1/2 & 1 \\ 0 & 3/2 \end{bmatrix}$
We see that the eigenvalues of this matrix are 1/2 and 3/2. Thus the phase portrait is that of a saddle and the origin is unstable.

(b) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
The eigenvalues of this matrix are $1 \pm i$ (check this), both of which have length $\sqrt{2} > 1$. Thus the phase portrait is that of a spiral outwards and the origin is unstable.

(c) $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$
The eigenvalues of this matrix are $1/2\pi i/2$ (check this), both of which have length $\frac{1}{2\sqrt{2}} < 1$. This the phase portrait is that of a spiral inwards and the origin is asymptotically stable.

(d) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
The eigenvalues of this matrix are 3 and 1. Thus the phase portrait has one direction which expands away from the origin and a perpendicular direction which is fixed. In particular, the origin is unstable.

11. (a) Let A be a 2×2 matrix. Show that the area of the parallelogram which is the image of $[0,1] \times [0,1]$ under A is $|\det A|$. (Hint: use the cross-product.)
Write A as

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

and let $v_1 = A(1,0) = (a,b)$ and $v_2 = A(0,1) = (c,d)$. The parallelogram in question has area $|v_1|\cdot|v_2|\sin\theta$, where θ is the angle between v_1 and v_2. Let $\tilde{v}_1 = (a,b,0)$ and $\tilde{v}_2 = (c,d,0)$. Then

$$|\det A| = |ad - bc| = |(0,0,ad-bc)| = |\tilde{v}_1 \times \tilde{v}_2| + |\tilde{v}_1| \cdot |\tilde{v}_2| \sin\theta = |v_1| \cdot |v_2| \sin\theta.$$

(b) Let A be a 3×3 matrix. Show that the area of the parallel-piped which is the image of $[0,1] \times [0,1] \times [0,1]$ under A is $|\det A|$. (Hint: use the cross-product and the dot product.)
This proof is much like the one in the last part. If we let v_1, v_2, v_3 be the columns of A, then the volume of the parallel-piped is $|v_1| \cdot |v_2| \cdot |v_3| \cos\phi \sin\theta$ where θ is the angle between v_2 and v_3 and ϕ is the angle between v_1 and the plane of v_2, v_3. Write A as

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Then

$$\begin{aligned} |\det A| &= |a(ei - fh) + b(fg - di) + c(dh - eg)| = |(a,b,c) \cdot (ei - fh, fg - di, dh - eg)| \\ &= |v_1 \cdot (v_2 \times v_3)| = |v_1| \cdot |v_2| \cdot |v_3| \cos\phi \sin\theta. \end{aligned}$$

(c) Let A be an $n \times n$ orthogonal matrix. Show that $\det A = \pm 1$ (argue geometrically).
That A is orthogonal means that A preserves lengths and angles. Thus the image of $[0,1]^n$ under A is a parallel-piped with sides of length 1 which meet at right angles. We can then conclude that the volume of this parallel-piped is 1, which in turn means that $\det A = \pm 1$.

12. For each of the matrices listed below, find its eigenvalues and eigenvectors, list the algebraic and geometric multiplicities of the eigenvalues, and find diagonalize the matrix (if possible).

(a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
The characteristic polynomial of A is

$$p(\lambda) = \det(\lambda \mathrm{Id} - \mathrm{A}) = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

Thus the eigenvalues are 1 and 3, each having algebraic multiplicity 1. This tells us immediately that the geometric multiplicity of each eigenvalues is also 1. It remains to find the eigenvectors. The $\lambda = 1$ eigenvector solves the equation $Av = v$, which (if we write v as (a, b)) gives us

$$2a + b = a,$$

or $a = -b$. The $\lambda = 3$ eigenvector solves the equation $Av = 3v$, which (if we write v as (a, b)) gives us

$$2a + b = 3a,$$

or $a = b$. Thus we have the two eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If we let S be the 2×2 matrix with v_1, v_2 as its columns then

$$S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
The characteristic polynomial of A is

$$p(\lambda) = \det(\lambda \mathrm{Id} - \mathrm{A}) = (\lambda - 1)^2.$$

Thus the only eigenvalue is 1 and it has algebraic multiplicity 2. Next we will find the eigenvectors of A. Any eigenvector v will have to solve the equation $Av = v$, which (if we write v and (a, b)) gives us

$$a + b = a.$$

Thus we must have $b = 0$ and to the only eigenvector (up to scale) we can find is

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus $\lambda = 1$ is an eigenvector of geometric multiplicity 1 (not 2, as we might expect). In particular, A is not diagonalizable (because it does not have a basis of eigenvectors).

13. Consider the regular octahedron with corners $p_{1,\pm} = (\pm 1, 0, 0)$, $p_{2,\pm} = (0, \pm 1, 0)$ and $p_{3,\pm} = (0, 0, \pm 1)$. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection which maps $p_{1,+}$ to $p_{1,-}$ and let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates $p_{1,+}$ to $p_{2,+}$. Note: you do not need to write down any matrices for this problem!

(a) Find the images of the other vertices of the octahedron under T_1.
T_1 is a reflection through the yz plane, so it fixes that plane. Therefore, $p_{2,+}$, $p_{2,-}$, $p_{3,+}$ and $p_{3,-}$ are all fixed. But $p_{1,+}$ and $p_{1,-}$ are interchanged by T_2.

(b) Find the images of the other vertices of the octahedron under T_2.
T_2 fixes the vertical axis (which is the axis of rotation), so $p_{3,+}$ and $p_{3,-}$ stay fixed. The rest of the vertices are permuted as follows:

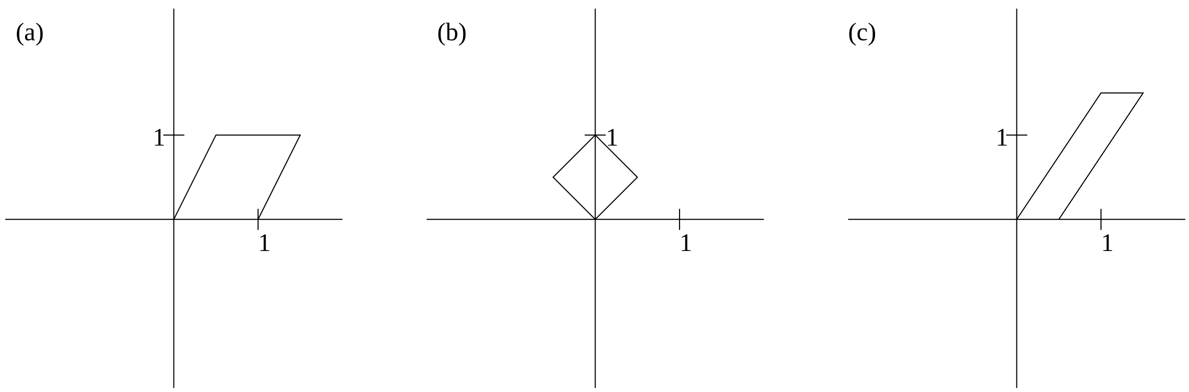
$$p_{1,+} \mapsto p_{2,+} \qquad p_{2,+} \mapsto p_{1,-} \qquad p_{1,-} \mapsto p_{2,-} \qquad p_{2,-} \mapsto p_{1,+}.$$

(c) Does there exist a positive integer k such that $T_1^k = \mathrm{Id}$? If there does, find it; if there does not, explain why.
Yes, $T_1^2 = \mathrm{Id}$. You can see this by applying T_1 twice to the list of vertices. Or you can see this because T_1 is a reflection.

(d) Does there exist a positive integer k such that $T_2^k = \mathrm{Id}$? If there does, find it; if there does not, explain why.
Yes, $T_2^4 = \mathrm{Id}$. Again, you can see this by chasing the vertices around four times (the permutation of $p_{1,\pm}, p_{2,\pm}$ written above is cyclic, so this is easy to do).

(e) Find the eigenvalues and eigenvectors of T_1. (Hint: think about this geometrically, without writing down any matrices.)
As noted above, T_1 fixes the yz plane. Thus for any vector v in the yz plane, we have $T_1 v = v$, and so v is an eigenvector of T_1 with eigenvalue 1. To find a basis for the $\lambda = 1$ eigenspace of T_1, pick your favorite basis of the yz plane, say $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$. Finally, we note that T_1 reverses the x direction, so (in particular) $T_1(1, 0, 0) = (-1, 0, 0)$. Thus $(1, 0, 0)$ is an eigenvector with eigenvalue -1.

14. Each of the following parallelograms represents the image of $[0, 1] \times [0, 1]$ under some linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. In each case, find $|\det T|$.



In each case, $|\det T|$ is the area of the parallelogram. The first parallelogram (labeled a) has base 1 and height 1; thus its area is 1. The second parallelogram (labeled b) is square with diagonal 1 (i.e. side length $\frac{1}{\sqrt{2}}$) and thus has area $\frac{1}{2}$. The third parallelogram (labeled c) has base $\frac{1}{2}$ and height $\frac{3}{2}$, and thus it has area $\frac{3}{4}$.

15. Consider the inner product of continuous functions on $[-\pi, \pi]$ defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt.$$

Also let V be the span of $v_0 = \frac{1}{\sqrt{2\pi}}$, $v_1 = \frac{1}{\sqrt{\pi}} \cos t$, $v_{-1} = \frac{1}{\sqrt{\pi}} \sin t$, $v_2 = \frac{1}{\sqrt{\pi}} \cos 2t$ and $v_{-2} = \frac{1}{\sqrt{\pi}} \sin 2t$.

(a) Show that $v_0, v_1, v_2, v_{-1}, v_{-2}$ form an orthonormal basis for V.
I will do the general computation for brevity. First we check

$$\|v_0\|^2 = \int_{-\pi}^{\pi} v_0^2(t)dt \int_{-\pi}^{\pi} \frac{1}{2\pi} dt = 1.$$

Next, for $j, k > 0$ and $j \neq k$ we check that

$$\langle v_j, v_k\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jt \cos kt dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((j-k)t) + \cos((j+k)t) dt = 0$$

and

$$\langle v_{-j}, v_{-k}\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin jt \sin kt dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((j-k)t) - \cos((j+k)t) dt = 0.$$

Also, for any j, k positive

$$\langle v_j, v_{-k}\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jt \sin kt dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin((j+k)t) - \sin((j-k)t) dt = 0.$$

Finally, we need to check that for $j > 0$

$$\langle v_j, v_j \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 jt dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 + \cos 2jt dt = 1$$

and

$$\langle v_{-j}, v_{-j} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 jt dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 - \cos 2jt dt = 1$$

(b) Let $T : V \to \mathbb{R}$ be the linear function defined by

$$T(f) = \langle f, g \rangle$$

where $g(t) = t^2$. Find the kernel of T. (Hint: you really don't need to do any computations. Think before you write!)
Notice that g is even while v_{-1} and v_{-2} are odd. Therefore

$$\int_{-\pi}^{\pi} g(t)v_{-1}(t)dt = 0 = \int_{-\pi}^{\pi} g(t)v_{-2}(t)dt$$

and both v_{-1} and v_{-2} are in the kernel of T. Also, $\int_{-\pi}^{\pi} g(t)v_0(t)dt = \int_{-\pi}^{\pi} t^2 dt > 0$ because the integrand is positive. Finally, it remains to check that

$$\int_{-\pi}^{\pi} t^2 \cos ntdt = \frac{t^2}{n}\sin nt\Big|_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} t\sin ntdt = -\frac{2}{n}[-\frac{t}{n}\cos nt\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos ntdt] = \frac{4\pi}{n^2}\cos n\pi \neq 0.$$

Thus the kernel of T is $\mathrm{span}\{\mathrm{v}_{-1}, \mathrm{v}_{-2}\}$.

(c) Find the function inside of V which lies closest to $f(t) = t$.
This function is given by the orthogonal projection of f onto V, which is

$$\mathrm{Proj_V f} = \mathrm{Proj_{v_{-2}} f} + \mathrm{Proj_{v_{-1}} f} + \mathrm{Proj_{v_0} f} + \mathrm{Proj_{v_{-2}} f} + \mathrm{Proj_{v_1} f} + \mathrm{Proj_{v_2} f} = \langle \mathrm{v_{-2}, f}\rangle \mathrm{v_{-2}} + \langle \mathrm{v_{-1}, f}\rangle \mathrm{v_{-1}} + \langle \mathrm{v_0, f}\rangle \mathrm{v_0} + \langle \mathrm{v_1, f}\rangle \mathrm{v_1} + \langle \mathrm{v_2, f}$$

However, f is an odd function while v_0, v_1, v_2 are all even. Thus $\langle v_0, f\rangle = 0$, $\langle v_1, f\rangle = 0$ and $\langle v_2, f\rangle = 0$. It remains to compute

$$\langle v_{-1}, f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} t\sin tdt = \frac{1}{\sqrt{\pi}}[-t\cos t|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} costdt] = 2\sqrt{\pi}$$

and

$$\langle v_{-2}, f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} t\sin 2tdt = \frac{1}{\sqrt{\pi}}[-\frac{t}{2}\cos 2t\Big|_{-\pi}^{\pi} + \frac{1}{2}\int_{-\pi}^{\pi}\cos 2tdt] = \sqrt{\pi}.$$

Thus the orthogonal projection of f onto V is

$$\mathrm{Proj_V f} = 2\sqrt{\pi}\,\mathrm{\sin t} + \sqrt{\pi}\,\mathrm{\sin 2t}.$$