

Solutions to Practice Problems

Math 2270

March 24, 2002

1. Consider the set of vectors $\{v_1, v_2, v_3\}$ in \mathbb{R}^3 where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

- (a) Are the vectors v_1, v_2, v_3 linearly independent? Do they form a basis for \mathbb{R}^3 ?

To check that $\{v_1, v_2, v_3\}$ are linearly independent, we look at the matrix with v_1, v_2, v_3 as its columns and row reduce it. Then we obtain

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

from which we can see that the row reduced echelon form of this matrix has three leading 1's. Thus $\{v_1, v_2, v_3\}$ are linearly independent.

Another way to see that v_1, v_2, v_3 are linearly independent is to go back to the definition. Suppose

$$av_1 + bv_2 + cv_3 = 0.$$

Looking at the first two components, this implies

$$\begin{cases} a + b - c = 0 \\ a - b + c = 0, \end{cases}$$

which in turn implies $a = 0$. Then looking at the second and third components, we see

$$\begin{cases} -b + c = 0 \\ b + c = 0, \end{cases}$$

which implies $b = 0$ and $c = 0$.

- (b) Let S be the 3×3 matrix whose columns are the vectors v_1, v_2, v_3 . What is the rank and the nullity of S ? Verify the Rank-Nullity theorem for S .

By the above computation, the rank of S is three. Because $\{v_1, v_2, v_3\}$ are linearly independent, the only solution to the equation

$$av_1 + bv_2 + cv_3 = 0$$

is the triple $a = 0, b = 0, c = 0$. This is another way of saying $\ker(S) = \{0\}$, and so the nullity of S is 0. The Rank-Nullity theorem says (in this case)

$$\text{rank}(S) + \text{nullity}(S) = 3,$$

which is indeed the case.

- (c) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation whose matrix representation (with respect to the standard basis) is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Compute the matrix of T in the basis $\{v_1, v_2, v_3\}$.

Recall that we can change coordinates via the formula

$$[T]_B = S^{-1}[T]S,$$

where S is the matrix of the previous part. First we need to compute S^{-1} , which is

$$S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$S^{-1}[T]S = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

- (d) Compute the rank and nullity of the matrix of T both in the standard basis and in the basis $\{v_1, v_2, v_3\}$. (You should obtain the same answer in both cases; can you explain why?)

First, why do you obtain the same answer for the rank and nullity, regardless of which basis you choose? The answer is that rank and nullity are properties which are independent of basis. To see this, observe that the rank is the dimension of the image of the linear transformation, and the nullity is the dimension of the kernel (vectors which get sent to 0).

It is easier to compute the rank and nullity of T in the new basis, so we will do that. First observe the matrix has a column of 0's, so the rank is at least two. But the remaining columns are linearly independent, which means that the rank is precisely two. Finally, by the Rank-Nullity theorem the nullity is one (because the sum must be three).

- (e) Apply the Gram-Schmidt process to $\{v_1, v_2, v_3\}$ to obtain an orthonormal set.

First we rescale v_1 to obtain a unit vector:

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Next we set

$$\tilde{u}_2 = v_2 - \text{Proj}_{u_1} v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -2/3 \\ 2/3 \end{pmatrix}.$$

Next we set

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \frac{2\sqrt{6}}{9} \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}.$$

Next we set

$$\tilde{u}_3 = v_3 - \text{Proj}_{u_1} v_3 - \text{Proj}_{u_2} v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 4/3 \\ -2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Finally, we set

$$u_3 = \frac{\tilde{u}_3}{\|\tilde{u}_3\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

2. Let P_2 be the space of polynomials of degree less than or equal to 2.

- (a) What is the dimension of P_2 ?

The dimension is 3. The easiest way to see this is to notice that you need precisely three parameters to write down a quadratic polynomial; a constant term, a linear term and a quadratic term.

- (b) Verify that $\{p_1 = t^2 + 1, p_2 = t^2 - 1, p_3 = t\}$ form a basis of P_2 .

We need to verify that if $ap_1 + bp_2 + cp_3 = 0$ then $a = 0, b = 0, c = 0$. Looking at the quadratic terms, we see that $a + b = 0$. But looking at the constant terms, we see that $a - b = 0$. Thus $a = 0, b = 0$. Finally, looking at the linear terms we see $c = 0$.

- (c) Write down the coordinates of $p = t^2 + t$ with respect to the basis $\{p_1, p_2, p_3\}$ listed above.

Notice we can write $t^2 = \frac{1}{2}(p_1 + p_2)$. Thus we can write the polynomial $p = t^2 + t$ as

$$p = \frac{1}{2}(p_1 + p_2) + p_3.$$

- (d) Consider the linear transformation $T : P_2 \rightarrow P_2$ given by $T(p) = p' - p$. Write down the matrix of T with respect to the basis $\{p_1, p_2, p_3\}$ listed above.

We need to see what the image of our basis is under this linear transformation. First note $p_1 \mapsto 2t - p_1 = 2p_3 - p_1$. Also, $p_2 \mapsto 2t - p_2 = 2p_3 - p_2$. Finally, $p_3 \mapsto 1 - p_3 = \frac{1}{2}(p_1 - p_2) - p_3$. Thus the matrix of this linear transformation is

$$[T] = \begin{bmatrix} -1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ 2 & 2 & -1 \end{bmatrix}.$$

- (e) Compute the image and kernel of T . (There are at least two ways to do this.)

Again, we can argue directly from the definition. If $T(ap_1 + bp_2 + cp_3) = aT(p_1) + bT(p_2) + cT(p_3) = 0$, then

$$a(-t^2 + 2t - 1) + b(-t^2 + 2t + 1) + c(-t + 1) = 0.$$

Looking at the quadratic terms gives us $a = -b$, while looking at the linear terms gives us $2a + 2b - c = 0$. Combining these gives us $c = 0$. Then looking at the constant terms (and using $c = 0$) gives us $a = b$. Thus we see that T has no kernel and the image of T is all of P_2 .

- (f) Verify that $\text{rank}(T) + \text{nullity}(T) = \dim(P_2)$.

From the previous part we see that the rank of T is 3 and the nullity of T is 0. They add up to 3, which is the dimension of P_2 .

3. Let V and W be abstract vector spaces.

- (a) Define what a subspace of V is.

A subspace of V is any collection $W \subset V$ such that $0 \in W$ and W is closed under addition and scalar multiplication. In other words, if $w_1, w_2 \in W$, then $w_1 + w_2 \in W$ and for any real number λ , $\lambda w_1 \in W$.

- (b) Define what a linear transformation from V to W is.

A linear transformation is any map

$$T : V \rightarrow W$$

such that, for any $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$ we have

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad T(\lambda v_1) = \lambda T(v_1).$$

- (c) If V is the space of all continuous function on $[-1, 1]$ (which is sometimes called $\mathcal{C}^0([-1, 1])$) and $T : V \rightarrow V$ is given by

$$T(f)(x) = f(-x),$$

is T a linear transformation? Explain your answer.

Yes, it satisfies the above properties.

- (d) If T is the space of continuous functions on $[-1, 1]$, and $T : V \rightarrow V$ is given by

$$T(f)(x) = (f(x))^2,$$

is T a linear transformation? Explain your answer.

No. For instance,

$$T(f + f) = (2f(x))^2 = 4(f(x))^2 \neq 2(f(x))^2 = (T(f) + T(f))(x).$$

- (e) Let V be the space of continuous functions of on $[-1, 1]$ and consider the linear transformation $T : V \rightarrow V$ given by

$$T(f)(x) = f(x) - \frac{1}{2} \int_{-1}^1 f(y) dy.$$

What is the kernel of T ? What is the nullity of T ?

If $T(f)(x) = 0$ for all x then

$$f(x) = \frac{1}{2} \int_{-1}^1 f(y) dy$$

for all x . In other words, f must be constant (and equal to its average value). Thus the kernel of T consists of the constant functions and the nullity is 1.

4. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

in \mathbb{R}^3 .

- (a) Are v_1 and v_2 linearly independent?

Yes, they are not scalar multiples of each other, so they are linearly independent.

(b) Find the orthogonal complement of $\{v_1, v_2\}$.

The orthogonal complement (written V^\perp) is all vectors orthogonal to both v_1 and v_2 . If $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is such a vector, then

$$\begin{cases} a + b + c = 0 \\ -a + b + c = 0. \end{cases}$$

From this we see $a = 0$ (taking the difference of the two equations) and $b + c = 0$ (taking the sum of the two equations). Thus the orthogonal complement is all the vectors of the form

$$\begin{pmatrix} 0 \\ b \\ -b \end{pmatrix}.$$

(c) Apply Gram-Schmidt to $\{v_1, v_2\}$. Call the set of vectors you obtain $\{u_1, u_2\}$.

First let

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Next let

$$\tilde{u}_2 = v_2 - \text{Proj}_{u_1} v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

Finally, we let

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \frac{2\sqrt{6}}{9} \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}.$$

(d) Complete $\{u_1, u_2\}$ to an orthonormal basis for \mathbb{R}^3 .

Now this part is easy; all we need to do is add in a unit vector in the direction of V^\perp . One such vector is

$$u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

5. Consider the square with corners $p_1 = (1, 1)$, $p_2 = (-1, 1)$, $p_3 = (-1, -1)$ and $p_4 = (1, -1)$.

(a) Let T be the rotation which sends p_1 to p_2 . find the image of the other three vertices of the square.

Under this rotation, we have $T(p_2) = p_3$, $T(p_3) = p_4$ and $T(p_4) = p_1$.

(b) Find the matrix (with respect to the standard basis) of T .

In the standard basis, the matrix representing T is

$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The easiest way to compute this is to notice T is a rotation by an angle of $\pi/2$ (in the counter-clockwise direction).

(c) Find the image of the vertices of the square under the map T^2 .

We have $T^2(p_1) = p_3$, $T^2(p_2) = p_4$, $T^2(p_3) = p_1$ and $T^2(p_4) = p_2$.

(d) Find a positive integer k such that $T^k = I$. (Hint: you can do this by just thinking of the images of the vertices.)

You can check that T^4 is the identity transformation. The easiest way to check this is to see (from the previous part) that $(T^2)^2$ is the identity.

6. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

in \mathbb{R}^3 and let Π be the span of v_1 and v_2 . Also, let

$$v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

- (a) Compare $\|v_1 + v_2\|^2$ to $\|v_1\|^2 + \|v_2\|^2$ without computing either quantity.

Note that $v_1 \cdot v_2 > 0$. Therefore, we have

$$\|v_1 + v_2\|^2 = (v_1 + v_2) \cdot (v_1 + v_2) = \|v_1\|^2 + \|v_2\|^2 + 2v_1 \cdot v_2 > \|v_1\|^2 + \|v_2\|^2.$$

- (b) Compare $|v_1 \cdot v_2|$ to $\|v_1\| \cdot \|v_2\|$ without computing either quantity.

Let θ be the angle between v_1 and v_2 . Notice that v_1 and v_2 are not parallel, which means θ is not an integer multiple of π . Therefore

$$|v_1 \cdot v_2| = |\cos \theta| \|v_1\| \cdot \|v_2\| < \|v_1\| \cdot \|v_2\|.$$

- (c) Compute the orthogonal project $\text{Proj}_{v_1} v$.

$$\text{Proj}_{v_1} v = \left(\frac{v_1 \cdot v}{\|v_1\|^2} \right) v_1 = \frac{1}{2} v_1.$$

- (d) Apply the Gram-Schmidt process to $\{v_1, v_2\}$.

First we let

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Next we let

$$u_2 = v_2 - \text{Proj}_{u_1} v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{u_1 \cdot v_2}{\|u_1\|} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Coincidentally, u_2 is a unit vector, so u_1, u_2 are an orthonormal basis for Π .

- (e) Compute the orthogonal projection of v onto the plane Π .

$$\text{Proj}_{\Pi} v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

7. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}.$$

- (a) For which θ is $\{v_1, v_2(\theta)\}$ a basis of \mathbb{R}^2 ?

To have v_1, v_2 a basis, we can't have them parallel. This condition is satisfied so long as $\theta \neq k\pi$ for any integer k . (Draw some pictures to convince yourself.)

- (b) for which θ is $\{v_1, v_2(\theta)\}$ an orthonormal basis of \mathbb{R}^2 ?

To make v_1, v_2 an orthonormal basis, we need $v_1 \perp v_2$ (they are always unit length). This is true for $\theta = \frac{2k+1}{2}\pi$ where k is any integer.

- (c) Compute the orthogonal projection of $v_2(\theta)$ onto v_1 .

$$\text{Proj}_{v_1} v_2 = \left(\frac{v_1 \cdot v_2}{\|v_1\|^2} \right) v_1 = \cos \theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ 0 \end{pmatrix}.$$

- (d) For which θ is it true that $\|v_1 + v_2(\theta)\|^2 > \|v_1\|^2 + \|v_2(\theta)\|^2$? (Hint: you do not need to compute lengths for this problem. Think about writing the length in terms of the dot product.)

This inequality is true when $v_1 \cdot v_2(\theta) > 0$, i.e. when the angle θ between v_1 and v_2 is between $-\pi/2$ and $\pi/2$.

- (e) For which θ is it true that $|v_1 \cdot v_2(\theta)| = \|v_1\| \cdot \|v_2(\theta)\|$? (Hint: again, you really don't need to compute anything.)

This is true when v_1 and v_2 are parallel, i.e. when $\theta = k\pi$ for any integer k .

8. Consider a linear map $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$.

- (a) Is it possible that T is onto? Is it possible that T is one to one?

T can be onto; $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$ is such an example. But T cannot be one to one. Because the dimension of the co-domain is 2, the rank of T is at most 2. Then by the Rank-Nullity theorem, the kernel of T must be at least 2-dimensional.

- (b) In general, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear and invertible, what can you say about m and n ? Answer the same question with the word “invertible” replaced by “one to one”.

If T is invertible, then we must have $m = n$. If T is one to one then we must have $n \leq m$.

- (c) What are the possible values for the rank of $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$?

The rank can be 0, 1 or 2. (This is because the rank is a non-negative integer not larger than the dimension of the domain and the dimension of the co-domain.)

- (d) What are the possible values for the nullity (dimension of the kernel) of $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$?

The nullity of T can be 4, 3, or 2 (apply the Rank-Nullity theorem to each case listed above).

- (e) If the rank of $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is 1, then what is its nullity?

By the Rank-Nullity theorem, the nullity must be 3.