Solutions to Practice Problems
Math 2270
January 28, 2002

1. Classify the following functions from $\mathbb{R}$ to $\mathbb{R}$ as linear, affine or other.
   (a) $f(x) = 2x + 1$ affine
   (b) $g(x) = x^2$ other
   (c) $h(x) = 3x$ linear

2. Consider the system of linear equations

$$\begin{aligned} x + y + z &= 2 \\ x - z &= 1 \\ x - y &= -1. \end{aligned}$$

   (a) Each of the linear equations yields a plane in $\mathbb{R}^3$. Find the normal for each plane and a point on each plane.
   The first equation corresponds to a plane through $(2, 0, 0)$ with normal $(1, 1, 1)$. The second equation corresponds to the plane through $(1, 0, 0)$ with normal $(1, 0, -1)$. The third equation corresponds to the plane through $(0, 1, 0)$ with normal $(1, -1, 0)$.
   (b) Write down the augmented matrix for this system.
   The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}.$$

   (c) Row reduce the matrix you wrote down in the previous part and find all solutions to this system. How many are there?
   After row reducing you should obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & -1/3 \end{bmatrix},$$

   which means that the *only* solution to this system is $x = 2/3$, $y = 5/3$ and $z = -1/3$.
   (d) Verify that the solution you found yields a vector perpendicular to the normal vectors to all three planes.
   This last part was perhaps worded poorly. What I mean is the following: if $p_1 = (2, 0, 0)$ is a point in the first plane and $x = (2/3, 5/3, -1/3)$ is the solution you found then $p_1 - x$ should be perpendicular to $n_1 = (1, 1, 1)$, which is the normal to the first plane. You can check this be computing the dot product $(p_1 - x) \cdot n_1$ and seeing that you get 0. All this is saying is that the point x lies in the first plane. You can also check that x lies in the other two planes with similar computations.

3. Consider a general linear map $A : \mathbb{R}^n \to \mathbb{R}^m$.
   (a) This map is given by matrix multiplication. What is the size of the matrix.
   The matrix A has m rows and n columns.
   (b) If A is one to one, what can you say about the relationship between n and m?
   If A is one to one then we must have $n \leq m$.
   (c) Now let $n = m = 2$ and change coordinates in $\mathbb{R}^2$ by rotating counterclockwise through an angle of $\pi/4$. How can you relate the matrix of A in the new coordinates to the matrix of A in the old (usual) coordinates?
   In this case we can shift from the old coordinates to the new coordinates by multiplying by the matrix

$$B := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

   The matrix B yields the linear transformation which is rotation by $\pi/4$ (counterclockwise). The linear transformation in the new coordinates can be written as multiplication by the matrix

$$C := B \cdot A \cdot B^{-1}.$$

   To see this, think about the following: first we rotate back by $-\pi/4$ to get the old coordinates, then we apply A and then we rotate by $\pi/4$ to get back to the new coordinates.

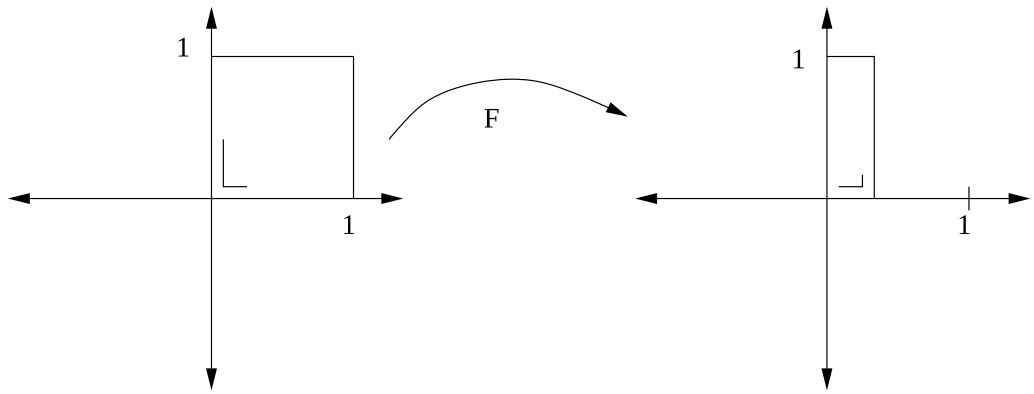
4. We will consider affine maps $\mathbb{R}^2 \to \mathbb{R}^2$ in this problem.

(a) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map

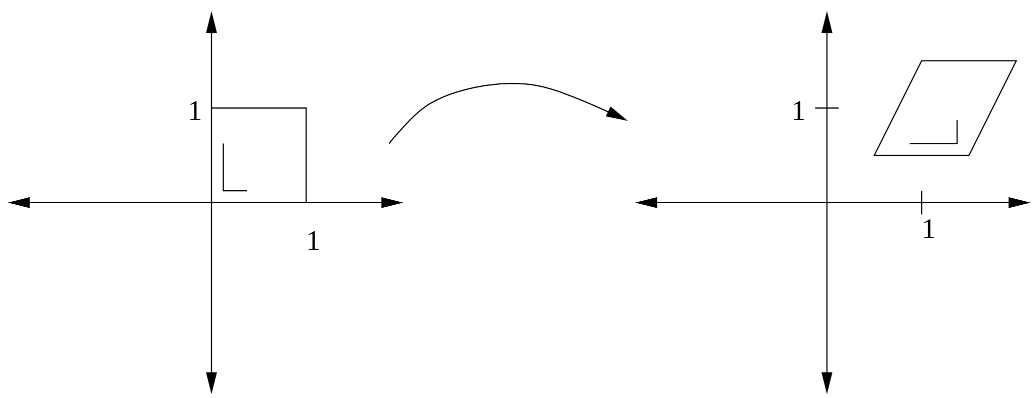
$$F(x) = \begin{bmatrix} 0 & -1/3 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}.$$

Draw a picture of the image of the unit square $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ under F. Be sure to indicate which corner is the image of $(0, 0)$ and the orientation of the image.

The linear part of F is a scaling in the vertical direction by a factor of $1/3$ followed by a rotation by $\pi/2$ (clockwise). Or, if you wish you can think of the linear part as a rotation by $\pi/2$ followed by a scaling by $1/3$ in the horizontal direction. Then we apply a horizontal translation by $1/3$ to the whole thing. The final picture is the following:



(b) Consider the parallelogram drawn below, which is the image of the unit square as indicated. Recover the affine map which generates this picture.



The easiest way to do this is to transform the parallelogram into a unit square, remembering what you do, and then work backwards. To transform the parallelogram into a unit square, first translate it by $(-3/2, -1/2)$, then rotate it by $-\pi/2$ and then shear by a factor of $1/2$ in the vertical direction.

Now we have to do all this in the reverse order to obtain the affine map: first shear by $-1/2$ vertically, which has the matrix representation

$$\begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}.$$

Next we have to rotate by $\pi/2$, which has the matrix representation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then finally we translate by $(3/2, 1/2)$. Putting this all together we see that the affine map is

$$F(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \cdot x + \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}.$$

5. Explain the relationship between a matrix A being invertible and the ability to solve the equation $Ax = b$ for x. In particular, remark on when you can solve this equation for any right hand side b, and on when you get a unique solution x to this equation.

The fact that one can find a solution x to the equation $Ax = b$ for any right hand side b says that A is onto. The fact that there is a unique solution x for any b (i.e. there is at most one solution for any given b) says that A is one to one. Putting these two properties together, we see that the equation $Ax = b$ has a unique solution x for any right hand side b if and only if A is invertible.

6. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$. Give a necessary and sufficient geometric condition for A to be invertible. (Hint: think about the image of the unit square under A.)

   The matrix A is invertible if and only the image of the unit square is not a line segment or a point; i.e. if and only if A does not collapse down any dimensions.

7. Let B be the matrix

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & 1 & 0. \end{bmatrix}$$

   (a) Find the inverse of B by row reducing.
   We start with the augmented matrix

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

   Which we can row reduce to

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{bmatrix}.$$

   Thus

$$B^{-1} = \begin{bmatrix} -1 & 1 & -2 \\ -1 & 1 & -1 \\ -2 & 1 & -2 \end{bmatrix}.$$

   (b) Multiply B by it's inverse (in either order) and verify that you get the identity matrix.
   You can check that both $B \cdot B^{-1}$ and $B^{-1} \cdot B$ are the 3×3 identity matrix.

   (c) Solve the equation $Bx = v$ where $v = (1, 0, 2)$ using the inverse you found for B.
   The solution x will have to have the form $x = B^{-1}v$ (why?), which we can compute is

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 1 & -1 \\ -2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix}.$$

8. Let A and B be 2×2 matrices. Is it true that $(A+B)^2 = A^2 + 2AB + B^2$? If it is true, explain why. Otherwise, find a counterexample and find a formula for $(A+B)^2$ in terms of A and B.

   Let

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   and let

$$B := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

   Then one can compute that

$$(A+B)^2 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

   while

$$A^2 + 2AB + B^2 = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}.$$

   The true rule is

$$(A+B)^2 = A^2 + AB + BA + B^2.$$

   You can prove this by writing out the product and being careful about the order in which you multiply matrices.