Practice Problems

Math 2270

April 25, 2002

These problems are in no particular order. I think they (together with the problems on the previous review sheets I've handed out) accurately reflect what will be on the exam. Of course, I haven't written the exam yet, so I could be wrong.

1. Classify the following maps as linear, affine or other.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - x$

   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$

   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x$

   (d) $f : P_2 \to P_2$, $P_2 =$ quadratic polynomials, $f(p)(t) = p'(t) - 2p(t)$.

   (e) $f : P_2 \to P_2$, $P_2 =$ quadratic polynomials, $f(p)(t) = p'(t) - 2p^2(t)$.

   (f) $f : P_2 \to P_2$, $P_2 =$ quadratic polynomials, $f(p)(t) = p'(t) - t^2$.

2. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ which has the matrix representation (in the standard basis)

$$[T] = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}.$$

   (a) Is T invertible? Explain your answer.

   (b) Find the kernel and range of T.

   (c) For which b can you solve the equation $T(v) = b$? Is the solution ever unique?

   (d) State the Rank-Nullity theorem and verify that it holds for T.

   (e) *Without computing any eigenvalues*, decide whether 0 is an eigenvalue of T. Explain your answer.

3. Consider the 2×2 matrix

$$T = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}.$$

   (a) For which values of θ is T invertible?

   (b) For which values of θ is T an orthogonal matrix?

   (c) Draw a picture of the parallelogram which is the image of $[0, 1] \times [0, 1]$ under T.

   (d) For which values of θ is the area of that parallelogram 1?

4. Consider the linear map $T : P_2 \to P_2$ given by

$$T(p)(t) = p(t) - p(-t),$$

   where P_2 is the collection of all polynomials of degree less than or equal to two.

   (a) Write down the matrix representation of T in the basis $B = \{t^2 + t, t^2 - t, t + 1\}$.

   (b) Find the kernel of T. (Hint: you do not need to use the matrix you computed in the previous part of this problem.)

   (c) Find the rank of T. (Hint: again, you do not need to use the matrix representation of T).

5. Consider a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$.

   (a) If T is one-to-one, what can you say about the relationship between n and m?

   (b) If T is onto, what can you say about the relationship between n and m?

   (c) In the particular case of $m = 4$ and $n = 2$, what are the possible pairings of the dimension of the kernel and the dimension of the range of T? For example, is it possible that the dimension of kernel of T is 3 while the dimension of the range of T is 2? List all possibilities.

6. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

and let V be the span of $\{v_1, v_2, v_3\}$.

(a) What is the dimension of V?

(b) Let T be the 4×3 matrix whose columns are v_1, v_2, v_3. What is the kernel of T?

(c) Apply the Gramm-Schmidt process to v_1, v_2, v_3 to obtain an orthonormal basis for V.

(d) Let

$$w = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

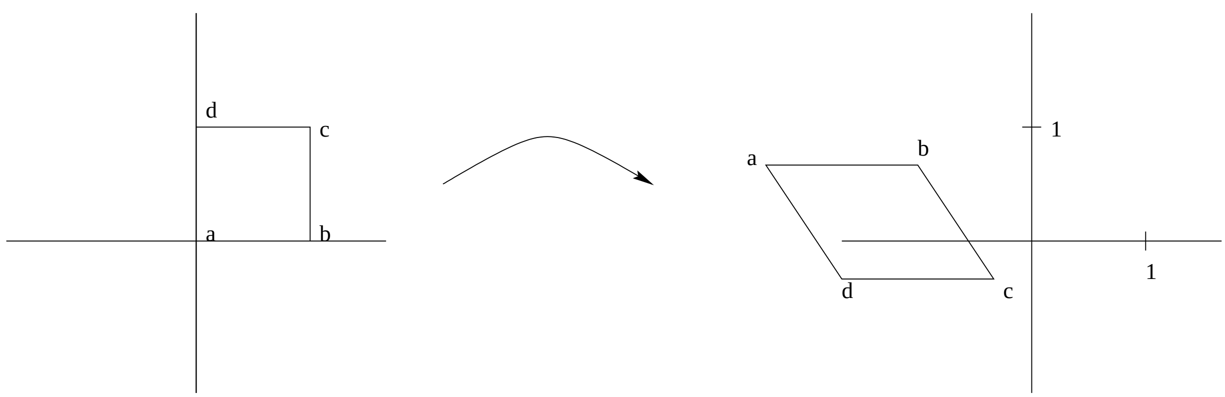
and compute the orthogonal projection of w onto V.

7. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map given by

$$T(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Draw the parallelogram which is the image of $[0,1] \times [0,1]$ under T, making sure to label the corners.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map which generates the following picture.



Find the matrix representation of this affine map (in the standard basis).

8. Let A be a 3×3 matrix with real entries.

(a) Is it possible that $i, 1+i, 1-i$ are the eigenvalues of A?

(b) Is it possible that $1, 1+i, 1+2i$ are the eigenvalues of A?

(c) Is it possible that $1, 1+i, 1-i$ are the eigenvalues of A?

(d) If A is orthogonal, is it possible that $1, 1+i, 1-i$ are the eigenvalues of A?

(e) How many nonreal eigenvalues can A possibly have? List all possibilities.

9. (a) Let A be an $n \times n$ matrix with real entries such that $A^t = A$. Also, let $\lambda_1 \neq \lambda_2$ be eigenvalues of A with eigenvectors v_1 and v_2 such that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Show that $v_1 \perp v_2$.

(b) If A is an $n \times n$ matrix such that $A^k = 0$ for some positive integer k, show that all the eigenvalues of A must be zero. (Note: this does not mean that A is the zero matrix. Can you think of some examples of matrices A such that $A \neq 0$ but $A^2 = 0$?)

(c) If A is an $n \times n$ matrix such that $A^k = \mathrm{Id}$ for some positive integer k, show that $|\lambda| = 1$ for all eigenvalues λ of A.

(d) Let A be an $n \times n$ orthogonal matrix (i.e. $A^{-1} = A^t$). Show that $|\lambda| = 1$ for all eigenvalues λ of A.

10. Consider the discrete dynamical system $x_{n+1} = Ax_n$ for some 2×2 matrix A. Notice that the origin is a fixed point of this system (i.e. the origin stays fixed). For each choice of matrix A listed below, classify the origin as unstable, stable or asymptotically stable and sketch a phase portrait for the system.

   (a) $A = \begin{bmatrix} 1/2 & 1 \\ 0 & 3/2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

11. (a) Let A be a 2×2 matrix. Show that the area of the parallelogram which is the image of $[0,1] \times [0,1]$ under A is $|\det A|$. (Hint: use the cross-product.)

   (b) Let A be a 3×3 matrix. Show that the area of the parallel-piped which is the image of $[0,1] \times [0,1] \times [0,1]$ under A is $|\det A|$. (Hint: use the cross-product and the dot product.)

   (c) Let A be an $n \times n$ orthogonal matrix. Show that $\det A = \pm 1$ (argue geometrically).

12. For each of the matrices listed below, find its eigenvalues and eigenvectors, list the algebraic and geometric multiplicities of the eigenvalues, and find diagonalize the matrix (if possible).

   (a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

13. Consider the regular octahedron with corners $p_{1,\pm} = (\pm 1, 0, 0)$, $p_{2,\pm} = (0, \pm 1, 0)$ and $p_{3,\pm} = (0, 0, \pm 1)$. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection which maps $p_{1,+}$ to $p_{1,-}$ and let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates $p_{1,+}$ to $p_{2,+}$. Note: you do not need to write down any matrices for this problem!

   (a) Find the images of the other vertices of the octahedron under T_1.

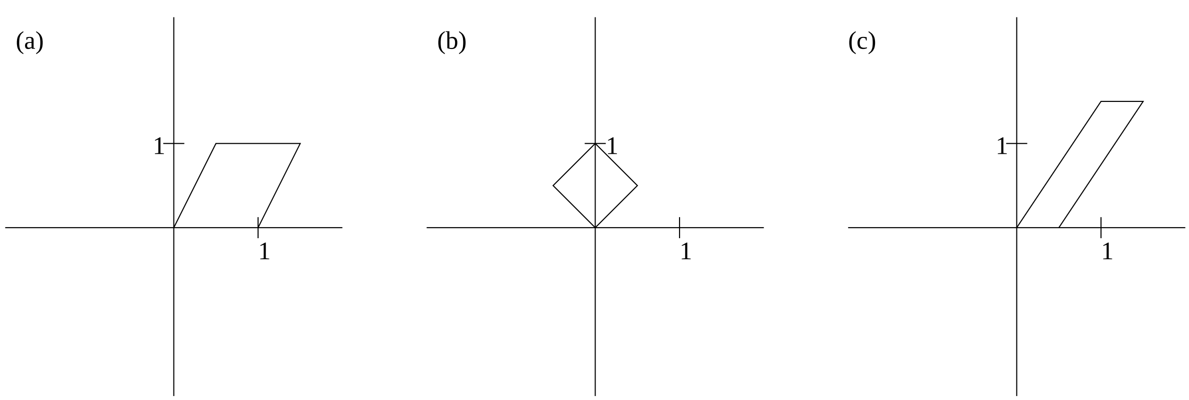
   (b) Find the images of the other vertices of the octahedron under T_2.

   (c) Does there exist a positive integer k such that $T_1^k = \mathrm{Id}$? If there does, find it; if there does not, explain why.

   (d) Does there exist a positive integer k such that $T_2^k = \mathrm{Id}$? If there does, find it; if there does not, explain why.

   (e) Find the eigenvalues and eigenvectors of T_1. (Hint: think about this geometrically, without writing down any matrices.)

14. Each of the following parallelograms represents the image of $[0,1] \times [0,1]$ under some linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. In each case, find $|\det T|$.



15. Consider the inner product of continuous functions on $[-\pi, \pi]$ defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)dt.$$

Also let V be the span of $v_0 = \frac{1}{\sqrt{2\pi}}$, $v_1 = \frac{1}{\sqrt{\pi}} \cos t$, $v_{-1} = \frac{1}{\sqrt{\pi}} \sin t$, $v_2 = \frac{1}{\sqrt{\pi}} \cos 2t$ and $v_{-2} = \frac{1}{\sqrt{\pi}} \sin 2t$.

(a) Show that $v_0, v_1, v_2, v_{-1}, v_{-2}$ form an orthonormal basis for V.

(b) Let $T : V \to \mathbb{R}$ be the linear function defined by

$$T(f) = \langle f, g \rangle$$

where $g(t) = t^2$. Find the kernel of T. (Hint: you really don't need to do any computations. Think before you write!)

(c) Find the function inside of V which lies closest to $f(t) = t$.