Practice Problems

Math 2270

March 20, 2002

These problems are in no particular order. I think they accurately reflect what will be on the exam. Of course, I haven't written the exam yet, so I could be wrong.

1. Consider the set of vectors $\{v_1, v_2, v_3\}$ in $\mathbb{R}^3$ where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) Are the vectors v_1, v_2, v_3 linearly independent? Do they form a basis for $\mathbb{R}^3$?

   (b) Let S be the 3×3 matrix whose columns are the vectors v_1, v_2, v_3. What is the rank and the nullity of S? Verify the Rank-Nullity theorem for S.

   (c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose matrix representation (with respect to the standard basis) is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

   Compute the matrix of T in the basis $\{v_1, v_2, v_3\}$.

   (d) Compute the rank and nullity of the matrix of T both in the standard basis and in the basis $\{v_1, v_2, v_3\}$. (You should obtain the same answer in both cases; can you explain why?)

   (e) Apply the Gram-Schmidt process to $\{v_1, v_2, v_3\}$ to obtain an orthonormal set.

2. Let P_2 be the space of polynomials of degree less than or equal to 2.

   (a) What is the dimension of P_2?

   (b) Verify that $\{p_1 = t^2 + 1, p_2 = t^2 - 1, p_3 = t\}$ form a basis of P_2.

   (c) Write down the coordinates of $p = t^2 + t$ with respect to the basis $\{p_1, p_2, p_3\}$ listed above.

   (d) Consider the linear transformation $T : P_2 \to P_2$ given by $T(p) = p' - p$. Write down the matrix of T with respect to to the basis $\{p_1, p_2, p_3\}$ listed above.

   (e) Compute the image and kernel of T. (There are at least two ways to do this.)

   (f) Verify that $\mathrm{rank(T)} + \mathrm{nullity(T)} = \mathrm{dim(P_2)}$.

3. Let V and W be abstract vector spaces.

   (a) Define what a subspace of V is.

   (b) Define what a linear transformation from V to W is.

   (c) If V is the space of all continuous function on $[-1, 1]$ (which is sometimes called $\mathcal{C}^0([-1, 1])$) and $T : V \to V$ is given by

$$T(f)(x) = f(-x),$$

   is T a linear transformation? Explain your answer.

   (d) If T is the space of continuous functions on $[-1, 1]$, and $T : V \to V$ is given by

$$T(f)(x) = (f(x))^2,$$

   is T a linear transformation? Explain your answer.

   (e) Let V be the space of continuous functions of on $[-1, 1]$ and consider the linear transformation $T : V \to V$ given by

$$T(f)(x) = f(x) - \frac{1}{2} \int_{-1}^{1} f(y) dy.$$

   What is the kernel of T? What is the nullity of T?

4. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^3$.

(a) Are v_1 and v_2 linearly independent?

(b) Find the orthogonal complement of $\{v_1, v_2\}$.

(c) Apply Gram-Schmidt to $\{v_1, v_2\}$. Call the set of vectors you obtain $\{u_1, u_2\}$.

(d) Complete $\{u_1, u_2\}$ to an orthonormal basis for $\mathbb{R}^3$.

5. Consider the square with corners $p_1 = (1, 1)$, $p_2 = (-1, 1)$, $p_3 = (-1, -1)$ and $p_4 = (1, -1)$.

(a) Let T be the rotation which sends p_1 to p_2. find the image of the other three vertices of the square.

(b) Find the matrix (with respect to the standard basis) of T.

(c) Find the image of the vertices of the square under the map T^2.

(d) Find a positive integer k such that $T^k = \mathrm{I}$. (Hint: you can do this by just thinking of the images of the vertices.)

6. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^3$ and let Π be the span of v_1 and v_2. Also, let

$$v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Compare $\|v_1 + v_2\|^2$ to $\|v_1\|^2 + \|v_2\|^2$ without computing either quantity.

(b) Compare $|v_1 \cdot v_2|$ to $\|v_1\| \cdot \|v_2\|$ without computing either quantity.

(c) Compute the orthogonal project $\mathrm{Proj}_{\mathrm{v}_1}\mathrm{v}$.

(d) Apply the Gram-Schmidt process to $\{v_1, v_2\}$.

(e) Compute the orthogonal projection of v onto the plane Π.

7. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

(a) For which θ is $\{v_1, v_2(\theta)\}$ a basis of $\mathbb{R}^2$?

(b) for which θ is $\{v_1, v_2(\theta)\}$ an orthonormal basis of $\mathbb{R}^2$?

(c) Compute the orthogonal projection of $v_2(\theta)$ onto v_1.

(d) For which θ is it true that $\|v_1 + v_2(\theta)\|^2 > \|v_1\|^2 + \|v_2(\theta)\|^2$? (Hint: you do not need to compute lengths for this problem. Think about writing the length in terms of the dot product.)

(e) For which θ is it true that $|v_1 \cdot v_2(\theta)| = \|v_1\| \cdot \|v_2(\theta)\|$? (Hint: again, you really don't need to compute anything.)

8. Consider a linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$.

(a) Is it possible that T is onto? Is it possible that T is one to one?

(b) In general, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and invertible, what can you say about m and n? Answer the same question with the word "invertible" replaced by "one to one".

(c) What are the possible values for the rank of $T : \mathbb{R}^4 \to \mathbb{R}^2$?

(d) What are the possible values for the nullity (dimension of the kernel) of $T : \mathbb{R}^4 \to \mathbb{R}^2$?

(e) If the rank of $T : \mathbb{R}^4 \to \mathbb{R}^2$ is 1, then what is its nullity?