Solutions to the Final Exam, Math 2270
May 6, 2002

1. Classify the following maps as linear, affine or other.

   (a) (2 points) $\beta : \mathbb{R}^2 \to \mathbb{R}$ given by $\beta(x, y) = 2x + y - xy$ other

   (b) (2 points) $T : C^\infty([-\pi, \pi]) \to \mathbb{R}$ given by $T(f) = \int_{-\pi}^{\pi} t^2 f(t)dt$ linear

   (c) (2 points) $\alpha : \mathbb{R} \to \mathbb{R}$ given by $\alpha(t) = 1 - t$ affine

2. Consider a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$.

   (a) (3 points) If T is one-to-one, what can you say about the relationship between m and n?
   $m \leq n$

   (b) (3 points) If T is onto, what can you say about the relationship between m and n?
   $m \geq n$

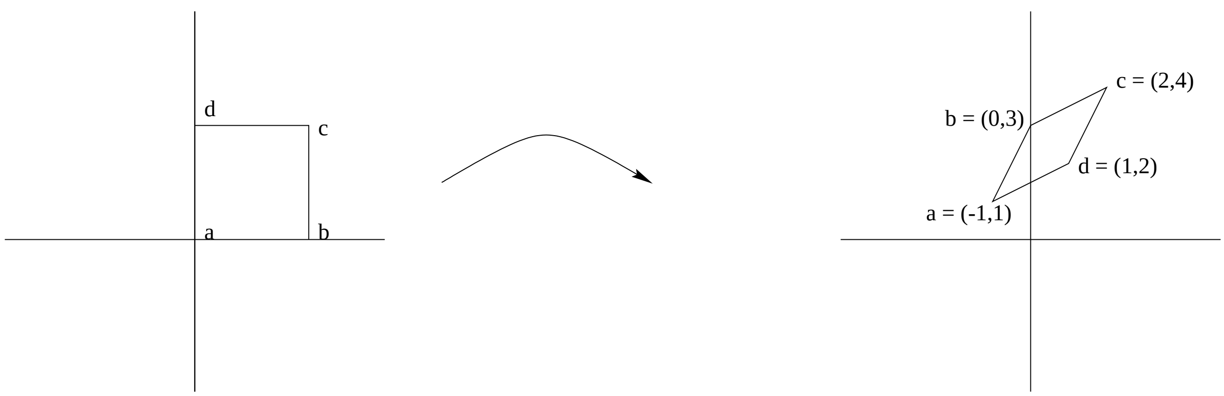
   (c) (3 points) If $m = 4$ and $n = 2$ (so that $T : \mathbb{R}^4 \to \mathbb{R}^2$) and the rank of T is 1, what is the dimension of the kernel of T? (Be sure to explain your answer.)
   We know that $\dim \mathbb{R}^4 = \dim \ker(\mathrm{T}) + \mathrm{rank}(\mathrm{T})$, so

$$\dim \ker(\mathrm{T}) = \dim \mathbb{R}^4 - \mathrm{rank}(\mathrm{T}) = 4 - 1 = 3.$$

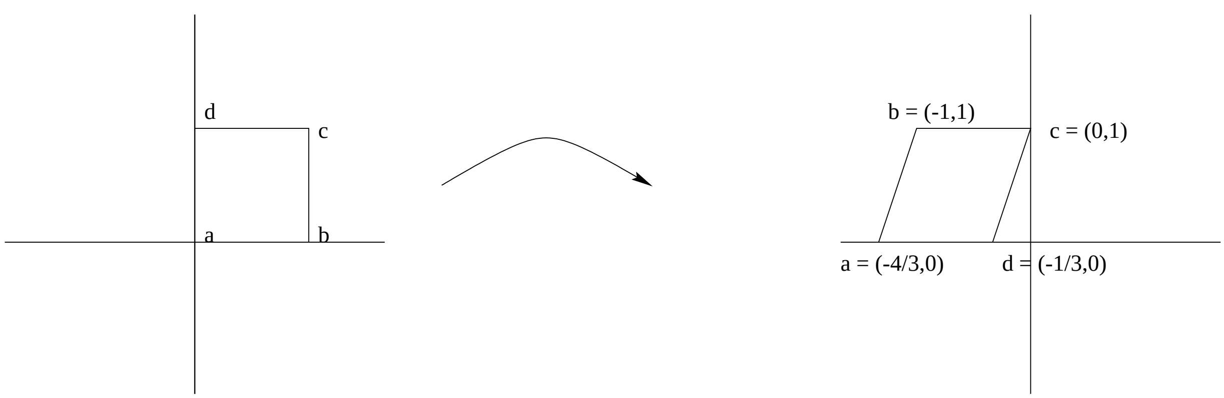
3. (a) (7 points) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T(v) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} v + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

   Sketch a picture of the parallelogram which is the image of $[0, 1] \times [0, 1]$ under T. Be sure to label the vertices of the parallelogram.



   (b) (7 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map which generates the following picture.



Find the matrix representation of this affine map (in the standard basis).

First, the translation part is given by the coordinates of the point matked a, which is $(-4/3, 0)$. The first column of the linear part is given by the difference between b and a, which is $(-1,1)-(-4/3,0) = (1/3,1)$. The second column is given by the difference between d and a, which is $(-1/3,0)-(-4/3,0) = (1,0)$. Thus the map T is given by

$$T(v) = \begin{bmatrix} 1/3 & 1 \\ 1 & 0 \end{bmatrix} v + \begin{pmatrix} -4/3 \\ 0 \end{pmatrix}.$$

4. Consider the vector space $V = \mathrm{span}\{1, \cos t, \sin t\}$ and let $T : V \to V$ be the linear transformation defined by

$$T(f)(t) = f(t) + f(-t).$$

(a) (4 points) What is the kernel of T?

In order that $T(f) = 0$ we must have $f(t) = -f(-t)$, in other words we need that f is odd. Of the functions listed (1, $\cos t$ and $\sin t$) only $\sin t$ is odd and the others are even. Thus

$$\mathrm{kerT} = \mathrm{span}\{\sin t\}.$$

(b) (3 points) What is the rank of T?

Notice $\dim V = 3$. Thus by the Rank-Nullity theorem

$$\mathrm{rank(T)} = \mathrm{dim\,V} - \mathrm{dim\,kerT} = 3 - 1 = 2.$$

(c) (4 points) Is 0 an eigenvalue of T? (Hint: you do not need to write down any matrices to answer this question.)

Yes, because the kernel of T is not just the zero vector. Recall that 0 is an eigenvalue if and only if $Tv = 0$ for some $v \neq 0$, which is the same thing as saying T has nontrivial kernel.

5. Consider the equilateral triangle with vertices $p_1 = (0,1)$, $p_2 = (-\sqrt{3}/2, -1/2)$ and $p_3 = (\sqrt{3}/2, -1/2)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation which sends p_1 to p_2. (Hint: you do not need to write down any matrices to do this problem. However, it may help to draw a picture.)

(a) (3 points) Find $T(p_2)$ and $T(p_3)$.

$T(p_2) = p_3$ and $T(p_3) = p_2$.

(b) (3 points) Is there as positive integer k such that $T^k = \mathrm{Id}$? If there is, find it. If there is not, explain why.

Notice that T is a rotation by $\frac{2\pi}{3}$. Thus T^k is a rotation by $\frac{2\pi k}{3}$ and, for $k = 3$, this is just a rotation by angle 2π, which leaves everything fixed. Thus $T^3 = \mathrm{Id}$.

(c) (4 points) Does T have any real eigenvalues? (Hint: think about the eigenvalue-eigenvector equation and what it means geometrically.)

No. Suppose $Tv = \lambda v$ for some real λ and $v \neq 0$. Then T would have to preserve the direction of v, which it can't because it is a rotation in the plane.

6. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and let $V = \mathrm{span}\{\mathrm{v_1, v_2, v_3}\}$.

(a) 3 points) What is the dimension of V?

Notice that $v_1 + v_2 = v_3$ and v_1, v_2 are not parallel. Thus $\dim V = 2$.

(b) (6 points) Apply the Gramm-Schmidt process to obtain an orthonormal basis for V.

Notice we only need to apply the Gramm-Schmidt process to v_1 and v_2, because $v_3 \in \mathrm{span}\{\mathrm{v_1, v_2}\}$. First,

$$u_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Next,

$$\tilde{u}_2 = v_2 - \text{Proj}_{\mathrm{u}_1}\mathrm{v}_2 = \mathrm{v}_2 - (\mathrm{u}_1 \cdot \mathrm{v}_2)\mathrm{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \\ 1/3 \\ 2/3 \end{pmatrix}.$$

Finally,

$$u_2 = \frac{\tilde{u}_2}{|\tilde{u}_2|} = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}.$$

(c) (6 points) Let

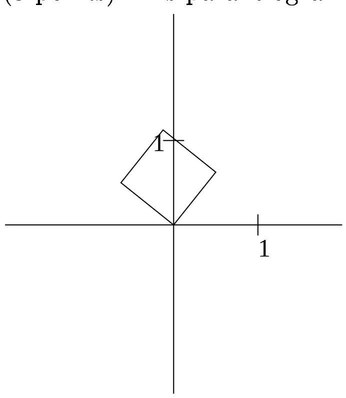
$$w = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and compute the orthogonal projection of w onto V.

$$\text{Proj}_V \mathrm{w} = \text{Proj}_{\mathrm{u}_1}\mathrm{w} + \text{Proj}_{\mathrm{u}_2}\mathrm{w} = (\mathrm{u}_1 \cdot \mathrm{w})\mathrm{u}_1 + (\mathrm{u}_2 \cdot \mathrm{w})\mathrm{u}_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \frac{4}{15} \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \\ 3 \\ 1 \end{pmatrix}.$$

7. Each of the following parallelograms represents the image of the unit square $[0,1] \times [0,1]$ under some linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$. In each case, find $|\det A|$.

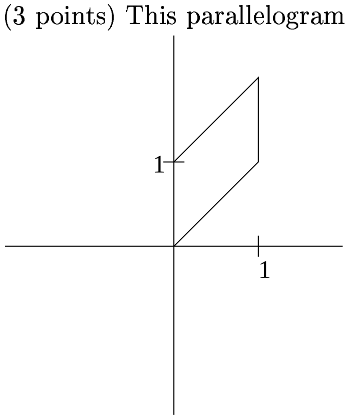
(a) (3 points) This parallelogram has vertices $(0,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}), (-\frac{\sqrt{3}}{2}, \frac{1}{2})$.



Notice that $(1/2, \sqrt{3}/2) \cdot (-\sqrt{3}/1, 1/2) = 0$, which tells you that the sides of this parallelogram meet at right angles. Therefore the parallelogram is in fact a rectangle, and the area of the parallelogram is the product of the side lengths. So

$$|\det A| = \text{Area} = |(1/2, /\text{sqrt}3/2)| \cdot |(-\sqrt{3}/2, 1/2)| = \sqrt{\frac{1}{4} + \frac{3}{4}} \cdot \sqrt{\frac{3}{4} + \frac{1}{4}} = 1.$$

(b) (3 points) This parallelogram has vertices $(0,0), (1,1), (1,2), (0,1)$.



This parallelogram has height 1 and width 1, therefore

$$|\det A| = \text{Area} = 1 \cdot 1 = 1.$$

8. Consider the discrete dynamical system $x_{n+1} = Ax_n$ for some 2×2 matrix A. Notice that the origin is a fixed point of this system regardless of A. For each choice of A listed below, classify the origin as unstable, stable or asymptotically stable and sketch a phase portrait of the system. Be sure to explain how you arrived at your answer.

   (a) (5 points) $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

   This matrix has eigenvalues 3 and 2, which means the phase portrait looks like it expands straight outward in every direction (with speeds varying between 2 and 3). Thus the origin is unstable.

   (b) (5 points) $A = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$

   This matrix has eigenvalues $\pm\frac{i}{2}$, which means the phase portrait looks like a spiral inward (the eigenvalues have magnitude $1/2 < 1$). Thus the origin is asymptotically stable.

   (c) (5 points) $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

   This matrix has eigenvalues $1/2$ and $3/2$, which means the phase portrait looks like a saddle. Thus the origin is unstable.

9. Let A be an $n \times n$ matrix with real entries.

   (a) (4 points) Show that the eigenvalues of A have to occur in complex-conjugate pairs.

   We know that the eigenvalues of A are roots of the polynomial $p_A(\lambda) = \det(\lambda\text{Id} - \text{A})$, which has real coefficients. By the Fundamental Theorem of Algebra, p_A has n complex roots, counted with multiplicity, which we can list as $\lambda_1 \dots \lambda_n$. Then we can rewrite p_A as

   $$p_A(\lambda) = k(\lambda - \lambda_1)\cdots(\lambda - \lambda_n).$$

   Now suppose the nonreal roots of p_A did not occur in conjugate pairs and plug in some $\lambda \in \mathbb{R}$ which is not a root. Because p_A has real coefficients, we must obtain a real number. But plugging λ into the above formula for p_A we find that $p_A(\lambda)$ is not real. This contradiction demonstrates that p_A must have real roots.

   (b) (4 points) Show that if $\lambda_+ = a + ib$ and $\lambda_- = a - ib$ are nonreal eigenvalues of A and $v + iw$ is an eigenvector associated to $\lambda_+ = a + ib$, then $v - iw$ is an eigenvector associated to $\lambda_- = a - ib$.

   We have that $A(v_iw) = (a + ib)(v + iw)$. Now take the complex conjugate of this equation and use the fact that A has real entries:

   $$A(v - iw) = (a - ib)(v - iw),$$

   which says that $v - iw)$ is an eigenvector with eigenvalue $(a - ib)$.

   (c) (6 points) Show that if A is symmetric ($A^t = A$) then A can only have real eigenvalues. (Hint: consider the dot product $\langle A(v + iw), v - iw\rangle$ and be sure to use the fact that A is symmetric.)

   We have that

   $$\langle A(v + iw), v - iw\rangle = \langle (a + ib)(v + iw), v - iw\rangle = (a + ib)\langle v + iw, v - iw\rangle = (a + ib)(|v|^2 + |w|^2).$$

   On the other hand, because A is symmetric we also have

   $$\langle A(v + iw), v - iw\rangle = \langle v + iwA(v - iw)\rangle = (a - ib)\langle v + iw, v - iw\rangle = (a - ib)(|v|^2 + |w|^2).$$

   We put these two together to obtain

   $$(a + ib)(|v|^2 + |w|^2) = (a - ib)(|v|^2 + |w|^2),$$

   which is only possible if $b = 0$.