Solutions to the Midterm Exam, Math 2270
March 28, 2002

1. Consider the square of side length $\sqrt{2}$ with corners at $p_1 = (1,0)$, $p_2 = (0,1)$, $p_3 = (-1,0)$ and $p_4 = (0,-1)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection which sends p_1 to p_3.

   (a) (4 points) Find the images of the other three corners of the square under this linear transformation.
   We have $T(p_3) = p_1$ and while p_2 and p_4 are fixed. (In fact, the line through p_2 and p_4 is the axis of the reflection, and so that entire line is fixed.) This linear trtansformation is a reflection through the y axis.

   (b) (4 points) Find the matrix which represents T (in the standard basis).
   The matrix representing T is
   $$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$
   The easiest way to see this is to compute the image of $(1,0)$ and $(0,1)$ (which you should have done in the previous part). This gives you the columns of the matrix.

   (c) (4 points) Is there a positive integer k such that T^k is the identity mapping? If there is, find k. If there is not, justify your claim.
   We have that
   $$T^2 = \text{Id}.$$
   This is true in general of any reflection. You can also see this by chasing the points around as in the first part of this problem. Or you could even see this by squaring the matrix representation for T.

2. Consider the vectors
   $$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) (4 points) Show that $B = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.
   There are several ways to do this, including row redcing the matrix with v_1, v_2, v_3 as the columns and seeing it has 3 leading ones. Another way to do this problem is the following. It suffices to show that v_1, v_2, v_3 is linearly independent, because there are three elements in this set and $\mathbb{R}^3$ is 3-dimensional. So suppose
   $$a_1 v_1 + a + 2v_2 + a_3 v_3 = 0$$
   for some numbers $a_1, a_2, a3$. This translates into the three equations
   $$\begin{cases} a_1 + a_2 &= 0 \\ a_3 &= 0 \\ a_1 - a_2 + a_3 &= 0. \end{cases}$$
   From the second equation, we see that
   $$a_1 + a_2 = 0 \qquad a_1 - a_2 = 0.$$
   The only solution to these equations is $a_1 = 0, a_2 = 0$. Thus v_1, v_2, v_3 are linearly independent.

   (b) (4 points) Let S be the 3×3 matrix whose columns are v_1, v_2 and v_3 (in that order). Explain why S is invertible and compute S^{-1}.
   S is invertible because its columns are linearly independent, which means it has rank 3, which means it's onto. By the Rank-Nullity theorem, the kernel of S is $\{0\}$, so S is one-to-one.
   The easiest way to compute S^{-1} is to row reduce, upon which you will obtain
   $$S^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

   (c) (5 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ have the matrix representation (in the standard basis)
   $$[T] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Compute the matrix representation of T in the basis B.
You could compute the matrix $[T]_B$ using the formula

$$[T]_B = S^{-1}[T]S.$$

However, I will compute $[T]_B$ directly below. First let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. Then we see that

$$T(v_1) = T(e_1) + T(e_3) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2v_1.$$

This means the first column of $[T]_B$ is

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly,

$$T(v_2) = T(e_1) - T(e_3) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = -2v_3$$

and

$$T(v_3) = T(e_2) + T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = v_1 + v_3.$$

Thus

$$[T]_B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

3. Let P_2 be the space of polynomials of degree at most two and let $T : P_2 \to P_2$ be given by

$$T(p)(t) = p(t) + p(-t).$$

(a) (4 points) What is the dimension of P_2? (Hint: try to think of a natural basis of P_2.)
To write down a second degree polynomial, you need to specify the constant term, the linear term and the quadratic term. Thus P_2 is three-dimensional.

(b) (4 points) What is the kernel of T? You do not need to write anything in coordinates to do this problem, but you may use coordinates if you wish.
If $T(p) = 0$ then $p(t) = -p(-t)$. Thus p can only have odd terms (i.e. the $p(t) = \sum_{k \text{ odd}} a_k t^k$). Because p has degree at most 2, p must be of the form $p(t) = at$, or

$$p(t) \in \text{Span}\{\text{t}\}.$$

Finally, we see that the kernel is one-dimensional and so the nullity is 1.

(c) (4 points) What is the rank of T? Again, you do not need to use any coordinates to do this problem.
By the Rank-Nullity theorem,

$$3 = \text{rank(T)} + \text{nullity(T)} = \text{rank(T)} + 1.$$

Thus the rank of T is 2.

4. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

and let $V = \text{span}\{\text{v}_1, \text{v}_2, \text{v}_3\}$.

(a) (4 points) What is the dimension of V?
Observe that $v_3 = v_1 + v_2$ and v_1, v_2 are not parallel. Thus the dimension of V is 2.

(b) (5 points) Apply the Gram-Schmidt process to $\{v_1, v_2, v_3\}$ to obtain an orthonormal basis of V.

Observe we only need to apply the Gram-Schmidt process to v_1 and v_2 (draw a picture). First we normalize v_1 to make it unit length. Let

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Next we let

$$\tilde{u}_2 = v_2 - \mathrm{Proj}_{\mathrm{u}_1} \mathrm{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1-1+1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -4/3 \\ 0 \\ 2/3 \end{pmatrix}.$$

Finally, we let

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

(c) (4 points) Let

$$u = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and compute the orthogonal projection of u onto the subspace V.

We have that

$$\mathrm{Proj}_{\mathrm{V}} \mathrm{u} = \mathrm{Proj}_{\mathrm{u}_1} \mathrm{u} + \mathrm{Proj}_{\mathrm{u}_2} \mathrm{u} = (\mathrm{u}_1 \cdot \mathrm{u})\mathrm{u}_1 + (\mathrm{u}_2 \cdot \mathrm{u})\mathrm{u}_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}.$$