

Solutions to the Midterm Exam, Math 2270
January 30, 2002

You have 50 minutes to complete this exam. It is a closed book, closed notes exam and there are 50 points possible. You may use a calculator that cannot do matrix computations, but a calculator will not be necessary. You may also leave all your numerical answers in natural form (e.g. write π instead of 3.14...). Show all your work to receive full credit.

Good luck.

1. Classify the following functions from \mathbb{R} to \mathbb{R} as linear, affine or other.

(a) (1 point) $f(x) = 1 - 2x$

affine

(b) (1 point) $g(x) = \sin x$

other

(c) (1 point) $h(x) = 3x$

linear

2. Consider the system of linear equations

$$\begin{aligned}x - y - z &= 1 \\2x + y - z &= 0 \\x - y + z &= -1.\end{aligned}$$

(a) (3 points) Each equation represents a plane in \mathbb{R}^3 . Find the normal vector of the plane corresponding to the first plane. The normal vector to the plane $x - y - z = 1$ is

$$n = (1, -1, -1).$$

(b) (2 points) Write down the augmented matrix for this system

The augmented matrix is

$$\left[\begin{array}{cccc} 1 & -1 & -1 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{array} \right].$$

(c) (6 points) Row reduce the matrix you wrote down in part (b) and find all solutions to this system. How many solutions did you find?

Upon row reducing, you will obtain

$$\left[\begin{array}{cccc} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \end{array} \right],$$

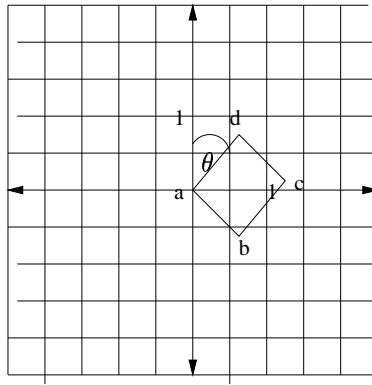
which means that the *only* solution is $x = -1/3$, $y = -1/3$ and $z = -1$.

3. Consider $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by

$$A(x) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} x,$$

where θ is some angle.

(a) (4 points) Draw a picture of the image of the unit square $\{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ under A . Be sure to label where the vertices go!



Here the original square $[0, 1]^2$ has its vertices labeled a, b, c, d proceeding counterclockwise around the square, with a at $(0, 0)$.

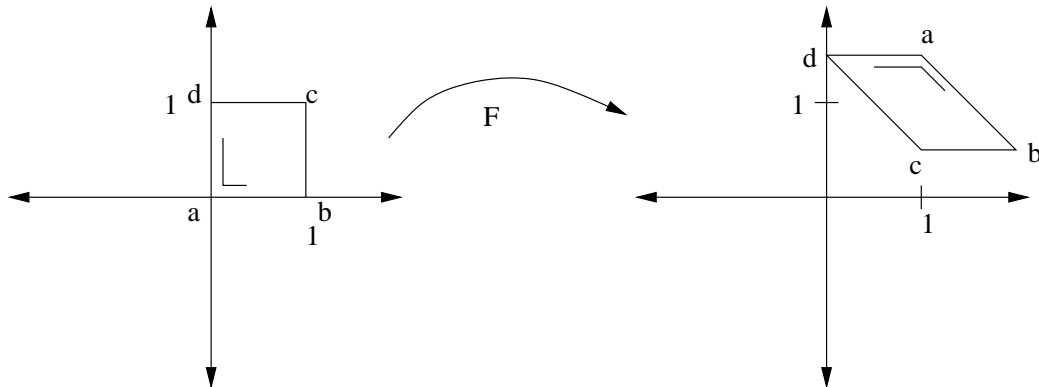
- (b) (4 points) Describe what A does as a linear map. (e.g. is it a shear, or a combination of a shear and a reflection?)

This linear map is a rotation by the angle $-\theta$. Alternatively, it is a rotation in the clockwise (i.e. negative) direction by the angle θ .

- (c) (4 points) Is A invertible? Explain your answer. (Note: you do not need to do any computations for this part.)

Yes, A is invertible. The image of the unit square is another square, not a line segment or a point.

4. (10 points) Consider the parallelogram drawn below, which is the image of the unit square under an affine map as indicated. Recover the affine map F which generates this picture.



The easiest way to do this is to work backwards. Then we will have to invert each step (which is not as hard as it sounds). First translate F to send the correct vertex to $(0,0)$; this is a translation by $b = (-1, -3/2)$. Next we reflect through the x axis; this is given by the matrix

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Conveniently, A_1 is its own inverse. Next we shear by -1 horizontally; this has the matrix

$$A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

You can check that you obtain A_2^{-1} by replacing the -1 with a 1 (this is easiest to check geometrically). Finally we rotate by $-\pi/2$, which has the matrix

$$A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

You can check that $A_3 = -A_3^{-1}$.

Now we can write F as

$$F(x) = A_1^{-1}A_2^{-1}A_3^{-1}x - b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}.$$

5. Let B be the 3×3 matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (6 points) Find the inverse of B by row reducing.

First set up the augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Upon row reducing this matrix you will obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

which means that

$$B^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) (4 points) Multiply B by the inverse you found in part (a) (in either order) and verify that you get the identity matrix.

You can check that both $B \cdot B^{-1}$ and $B^{-1} \cdot B$ are the 3×3 identity matrix.

(c) (4 points) Solve the matrix equation $Bx = v$ with $v = (0, 2, -1)$ using the inverse you found in part (a).

The solution x will have the form $x = B^{-1}v$ (why?), which we can compute is

$$x = B^{-1}v = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -1 \end{bmatrix}.$$