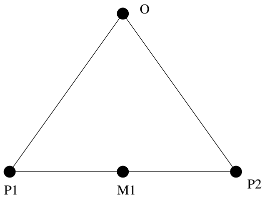
Midterm Exam

Math 223

March 28, 2005

1. (10 points) Find the area and perimeter of the inscribed regular hexagon (6-gon) in a unit circle. (Hint: use trigonometric functions. You might recall that $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/6) = \sqrt{3}/2$, and $\sin(\pi/6) = 1/2$.)

   First let γ be the unit circle, with center O, and let $P_1, \ldots, P_6$ be the vertices of a regular inscribed hexagon in γ, with P_i and P_{i+1} adjcent. The figure below shows $\triangle P_1OP_2$.



   Note that $|OP_i| = 1$ for $i = 1, \ldots 6$ and $|P_iP_{i+1}| = |P_jP_{j+1}|$. Thus the triangles $\triangle P_iOP_{i+1}$ are all congruent by SSS, so $\angle P_iOP_{i+1} \simeq \angle P_jOP_{j+1}$ for any i and j. This implies each angle has measure 60° (because all six have to sum to 360°, and they all have equal measure). Moreover, because $OP_i \simeq OP_{i+1}$, we have

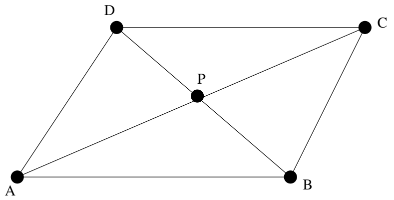
$$(\angle OP_iP_{i+1})^\circ = (\angle OP_{i+1}P_i)^\circ = \frac{1}{2}(120^\circ) = 60^\circ.$$

   We conclude that each triangle $\triangle P_iOP_{i+1}$ is an equilateral triangle with sidelength 1. Therefore the hexagon has perimeter 6.

   Next we compute the area. Let M_i be the midpoint of P_iP_{i+1}. Then $\triangle OP_iM_i \simeq OP_{i+1}M_i$ by SSS, so $\angle OM_iP_i$ is a right angles and $(\angle OP_iM_i)^\circ = 30^\circ = (\angle OP_{i+1}M_i)^\circ$. Therefore the height of $\triangle P_iOP_{i+1}$ is $\cos(30^\circ) = \sqrt{3}/2$. We already know that the base of this triangle is 1, so we have area$(\triangle P_iOP_{i+1} = (1/2)(1)(\sqrt{3}/2) = \sqrt{3}/4$. Thus the area of the hexagon is $3\sqrt{3}/2$.

2. (10 points) Let $\square ABCD$ be a quadilateral in the Euclidean plane with diagonals AC and BD crossing at the point P. Show that if $AP \simeq PC$ and $BP \simeq PD$ then $\square ABCD$ is a parallelogram, *i.e.* that opposite sides are parallel and have equal length. (Hint: congruent triangles. You may use Euclid V.)

   Here is a pitcure:



   We know that $\angle APB \simeq \angle DPC$ and $\angle APD \simeq \angle BPC$, because each is a pair of vertical angles. Therefore, $\triangle APD \simeq \triangle CPB$ and $\triangle APB \simeq \triangle CPD$ by SAS (note the ordering of the vertices). This implies $AD \simeq CB$ and $AB \simeq CD$.

   It remains to show that the opposite sides are parallel; there are (at least) two different proofs. First look at angle sums.

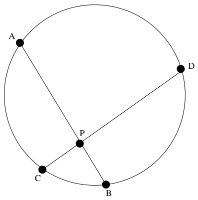
$$\begin{aligned}(\angle BAP)^\circ + (\angle PAD)^\circ + (\angle PDA)^\circ + (\angle CDP)^\circ &= (\angle BAP)^\circ + (\angle CDP)^\circ + 180^\circ - (\angle APD)^\circ \\ &= (\angle DCP)^\circ + (\angle CDP)^\circ + 180^\circ - (\angle APD)^\circ \\ &= 180^\circ - (\angle DPC)^\circ + 180^\circ - (\angle APD)^\circ = 180^\circ\end{aligned}$$

   Here we have used the fact that the angle sum of a triangle is 180° and Then by Euclid V, AB is parallel to DC. A similar argument shows $(\angle ABP)^\circ + (\angle PBC)^\circ + (\angle BCP)^\circ + (\angle PCD)^\circ = 180^\circ$, and so AD is parallel to BC.

Here's another argument. We know that $\angle BAP \simeq \angle DCP$, so by the alternate interior angles theorem AB is parallel to DC. Similarly, BC is parallel to AD. However, you have to be careful here. For instance, saying $\angle Pda \simeq \angle PBC$ doesn't work. (Why?).

3. (10 points) Suppose two chords AB and CD of a circle γ intersect at a point P. Prove that $|AP| \cdot |PB| = |CP| \cdot |PD|$, where $|AP|$ denotes the length of the segment AP. (It might help to draw a picture.)

   Here's the picture you might have drawn.



   Some people drew a picture with P outside the circle, which really doesn't happen because the chords lie inside the circle. However, the argument those people gave is essentially the same as the argument below, so I decided to give those people credit.

   Note that $\angle CAB$ and $\angle BDC$ are inscribed angles on γ which span the same arc, so $\angle CAB \simeq \angle BDC$. Similarly, $\angle ACD \simeq \angle DBA$. Also, $\angle APC \simeq \angle DPB$ because they're vertical angles. Thus $\triangle APC \sim \triangle DPB$ by AAA, and

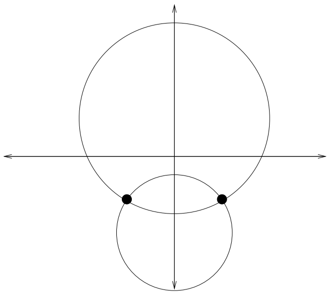
$$\frac{|AP|}{|PD|} = \frac{|CP|}{|PB|}.$$

   We can rearrange this to read $|AP||PB| = |CP||PD|$.

4. Let γ_1 be the circle centered at $(0, 10)$ of radius 25 and let γ_2 be the circle centered at $(0, -20)$ of radius 15. Also let P be in $\gamma_1 \cap \gamma_2$.

   (a) (5 points) How many possibilities are there for P? Can you list them? (You might want to draw a picture, labeling γ_1, γ_2, and the possibilities for P.)

   Here's a picture, with the two possibilities for P marked by black dots.



   We will see that there are two possibilities for P. Note that γ_1 and γ_2 are given in cartesian coordinates as

$$\gamma_1 : x^2 + (y-10)^2 = 625 \qquad \gamma_2 : x^2 + (y+20)^2 = 225.$$

   Subtracting these two equations gives us

$$(y-10)^2 - (y+20)^2 = 400 \Leftrightarrow y = -\frac{35}{3}.$$

   Plugging this back into either equations, we get an expression for x involing a square root:

$$x = \pm\sqrt{625 - (65/3)^2} \approx \pm 12.47,$$

   which gives us cartesian for both possibilities of P.

(b) (5 points) Is there another circle γ_3 such that P would be uniquely determined if it lies on all three circles. If there is, find one such circle. If there is not, explain your reasoning.

Yes, take a circle with a center not on the y-axis, which passes through P. For example, we can take the center of γ_3 to be $(10, 0)$, with the appropriate radius.

The easiest way to see that this is the right condition is to look for the conditions under which γ_3 contains both the possibilities for P. Then γ_3 has the equation

$$(x - x')^2 + (y - y')^2 = r_3^2,$$

where (x', y') is the center of γ_3 and r_3 is its radius. The fact that γ contains both possibilities for P, which we label $(\pm x^*, y^*)$, means that

$$(x^* - x')^2 + (y^* - y')^2 = r_3^2 = (-x^* - x')^2 + (y^* - y')^2 \Leftrightarrow x' = -x' \Leftrightarrow x' = 0.$$

Thus the only case where γ_3 contains both possibilities for P is when its center lies on the y-axis.

5. Let α be an angle such that $m(\angle\alpha) = (\alpha)^\circ < 90^\circ$.

(a) (3 points) State the definition of $\sin\alpha$ in terms of right triangles.

Let $\triangle ABC$ be a right triangle, with one of the acute angles congruent to α. Also let AB be the hypotenuse and BC the side opposite α. Then

$$\sin(\alpha) = \frac{|BC|}{|AB|}.$$

(b) (7 points) Show that $\sin\alpha$ is well-defined. In other words, show that $\sin\alpha$ does not depend on the right triangle you used in formulating the definition above, so long as it contains an angle congruent to α.

Let $\triangle ABC$ and $\triangle DEF$ be two right triangles as in the definition of $\sin\alpha$, with AB and DE opposite the right angle, and BC and EF opposite α. Then

$$(\angle ABC)^\circ = 180^\circ - 90^\circ - (\angle\alpha)^\circ = (\angle DEF)^\circ,$$

so $\angle ABC \simeq \angle DEF$. Thus $\triangle ABC \sim \triangle DEF$ by AAA, and

$$\frac{|BC|}{|AB|} = \frac{|EF|}{|DE|},$$

which completes the proof.