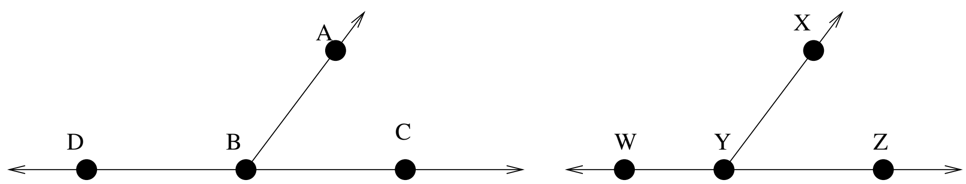
Solutions to the Midterm Exam

Math 223

Feb. 16, 2005

1. (10 points) Prove that supplementary angles to congruent angles are congruent. More precisely, if in the following figure $\angle ABC \simeq \angle XYZ$ then $\angle ABD \simeq \angle XYW$. (Hint: congruent triangles. Also notice that you can move A and C, for instance, along the ray without changing the angle.)



First notice that we can make $AB \simeq XY$ and $BC \simeq YZ$, so $\triangle ABC \simeq \triangle XYZ$ by SAS. Then we can make $DB \simeq WY$, which implies $DC \simeq WZ$. We also know $AC \simeq XZ$ and $\angle ACD \simeq \angle XZW$, so $\triangle ACD \simeq \triangle XZW$ by SAS. Finally, this tells us $BD \simeq YW$, $\angle BDA \simeq \angle YWX$, and $DA \simeq WX$, so $\triangle DBA \simeq \triangle YWX$ by SAS. So $\angle ABD \simeq \angle XYW$, because they are corresponding parts of congruent triangles.

2. Consider the following two incidence geometries:

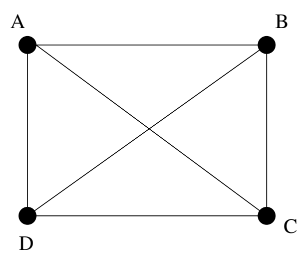


Figure 1: Geometry A, with lines $\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}, \{A, C\}, \{B, D\}$

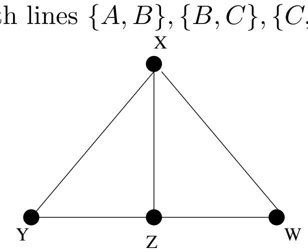


Figure 2: Geometry B, with lines $\{X, Y\}, \{X, Z\}, \{X, W\}, \{Y, Z, W\}$

(a) (5 points) Are these two incidence geometries isomorphic? Explain your answer.

No. Geometry A has 6 lines while Geometry B has 4 lines, so there cannot be a one-to-one pairing of lines.

(b) (5 points) In Geometry B, list all the lines through W which are parallel to $\{X, Z\}$.

All the lines through W are also incident to either X or Z, so there are no parallel lines.

3. (10 points) Justify the steps in showing that one can construct an angle measuring 75° using a straightedge and compass.

Notice that $75 = 45 + 30$, so it suffices to construct a 45° angle and a 30° angle.

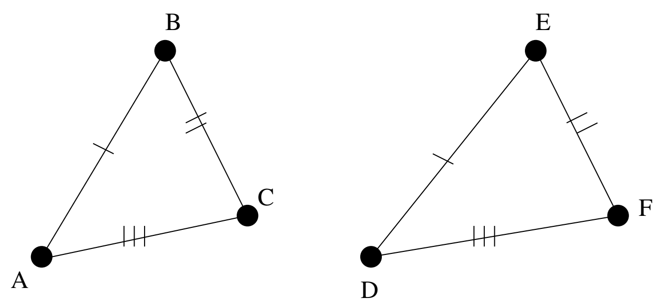
45°: start with a straight line and mark a point A on it, then mark two points, B and C, on the line which have equal distance to A. Next mark two circular arcs, centered at B and C, both of which have the same radius $r > |BA|$. These circular arcs intersect at a point D; connect D and A with a straight line segment. Now $\angle DAB$ is a right angle (because $\triangle ABD \simeq \triangle ACD$ by SSS). If you mark a point E on the first straight line so that B is between A and E and $|AE| = |AD|$, then $(\angle ADE)^\circ = 45^\circ = (\angle AED)^\circ$, because the two angles must sum to 90° and they are congruent.

30°: start with a point A and a circle of radius r centered at A, which you can draw using the compass. Choose any point B on the circle, and use the compass to draw a circle centered at B of the same radius r, so this new circle passes through A. Call one of the intersection points of the two circles C, and use the straightedge to draw the triangle $\triangle ABC$; all the angles of $\triangle ABC$ are $60°$, because $\triangle ABC$ is an equilateral triangle. Now we pick one angle, say $\angle BAC$ to bisect. Use the compass to mark points $D \in AB$ and $E \in AC$ such that $|AD| = |AE|$. Now use the compass to draw circular arcs centered at D and E, of the same radius ρ, which intersect at a point F. Then $\triangle ADF \simeq \triangle AEF$ by SSS, so $\angle BAF \simeq \angle CAF$. Then the ray $\vec{AF}$ bisects $\angle BAC$, so $(\angle BAF)^\circ = 30°$.

4. (10 points) Euclid wrote, *If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.*

   Draw a sketch illustrating this proposition and restate it in modern language. What does it mean?

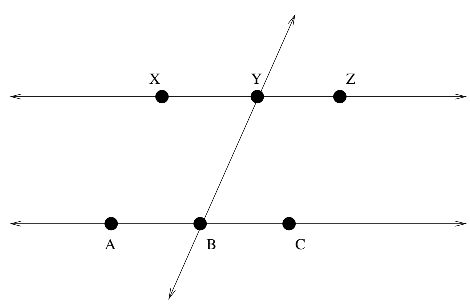
   Here's the picture:



   This statement is better know as the side-side-side congruence rule for triangles. It says that if three pair of sides of two triangles are congruent then the two triangles are congruent.

5. (10 points) Prove that Euclid V implies the converse to the alternate interior angles theorem. (Hint: argue by contradiction.)

   We start with two parallel lines, l_1 and l_2, which are cut by a transversal t. We want to show the alternate interior angles $\angle ABY$ and $\angle ZYB$ are congruent. Here's a picture:



   Let's suppose they're not congruent and derive a contradiction. If $\angle ABY \not\simeq \angle ZYB$ then either $(\angle ABY)^\circ > (\angle ZYB)^\circ$ or $(\angle ABY)^\circ < (\angle ZYB)^\circ$. In the case that $(\angle ABY)^\circ > (\angle ZYB)^\circ$, we have

$$(\angle ZYB)^\circ + (\angle CBY)^\circ = (\angle ZBY)^\circ + 180° - (\angle ABY)^\circ < 180°,$$

   so Euclid V implies l_1 and l_2 intersect. This contradicts the fact that l_1 and l_2 are parallel. Similarly, in the case that $(\angle ABY)^\circ < (\angle ZYB)^\circ$, we get $(\angle ABY)^\circ + (\angle XYB)^\circ < 180°$, and so l_1 and l_2 again intersect by Euclid V. The only possibility that does not lead to a contradiction is $\angle ABY \simeq \angle ZYB$, so that's what must occur.