Practice Problems

Math 223

April 20, 2005

These problems are in no particular order.

1. Prove that Euclid V is equivalent to the angle sum of any triangle being $180°$.

2. Consider the triangle $\triangle ABC$, and let $\vec{BD}$ bisect the angle $\angle ABC$. Show that $\vec{BD} \perp AC$ if and only if $AB \simeq BC$.

3. Recall that a convex polygon P satisfies the property that every line segment joining two points inside P remains inside P. Let P be a convex polygon with vertices $X_1, \ldots X_n$ (numbered in order), and let l be a line passing through the side X_1X_2, but not through one of the vertices. Show that l must pass through another side of P. (Hint: use Pasch's axiom for triangles and induction.)

4. Let two circles c_1 and c_2 intersect at points P and Q, and let AP and BP be diameters of c_1 and c_2 (respectively). Show that AB passes through the other intersection point Q.

5. Let c_1 and c_2 be mutually tangent circles, meeting at T. Show that the line connecting the centers O_1 and O_2 passes through T.

6. Show that stereographic projection preserves angles. More precisely, if $\Phi : S^2 \backslash \{N\} \to \mathbb{R}^2$ is stereographic projection and c_1 and c_2 circles through N on S^2, meeting at a point P, then the image lines $l_1 = \Phi(c_1)$ and $l_2 = \Phi(c_2)$ meet at the same angle at $Q = \Phi(P)$. (Hint: you don't need to use any formulas.)

7. Prove the law of cosines:
$$c^2 = a^2 + b^2 - 2ab\cos(\theta),$$
where a, b, c are side lengths of a triangle and θ is the angle opposite c.

8. Let f and g be isometries. Show that $f \circ g$ is also an isometry.

9. Recall that a regular n-gon is an n-sided polygon such that all its sides and interior angles are congruent. Show that a regular n-gon has n lines of reflection symmetry. (Hint: look at the bisectors of the interior angles and the perpendicular bisectors of the sides.)

10. Let r_l be reflection through the line l and r_m reflection through the line m. Show that $r_m \circ r_l \circ r_m = r_{l'}$, where l' is the reflection of l through m. (Hint: let A and B be distinct points on l. Show that $r_m(A)$ and $r_m(B)$ are fixed points of $r_m \circ r_l \circ r_m$.)

11. Let r_m be reflection through the line m. Show that r_m maps the line l to itself (i.e. l is invariant under r_m) if and only if $l \perp m$.

12. Let Γ be the lattice consisting of the points
$$(n, m), m \text{ even}, \qquad (n + \frac{1}{2}, m), m \text{ odd}.$$
Find two non-parallel translations which preserve Γ.

13. Let $T = r_2 \circ r_1$ be a translation, where r_1 and r_2 are reflections. Find a formula for T^{-1} in terms of r_1 and r_2.

14. Let T be a translation which is not the identity. Show that if l_1 and l_2 are two lines which are invariant under T, then $l_1 \| l_2$. (Recall that a line l is invariant under T if and only is T maps l to itself.)

15. Let R be a rotation with fixed point O, which is not the identity. Show that if l is a line which does not pass through O, then R does not map l to itself.

16. Let R be a rotation. Show that if R has a fixed line then R is rotation by $180°$.

17. A Saccheri quadrilateral $\square ABCD$ satisfies $AB \simeq CD$ and $(\angle ABC)^\circ = 90^\circ = (\angle BCD)^\circ$. Also, let M be the midpoint of BC and M' the midpoint of DA.

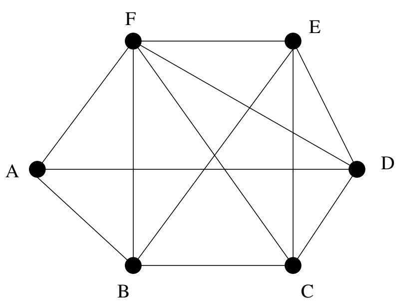
    (a) Show that in both the Euclidean and hyperbolic plane, $\angle BAD \simeq \angle CDA$. (Hint: congruent triangles.)

    (b) Show that in both the Euclidean and hyperbolic plane, MM' meets both the sides BC and DA at right angles. (Hint: more congruent triangles.)

    (c) Show that in the hyperbolic plane, $(\angle BAD)^\circ < 90^\circ$. (Hint: angle sum of a triangle.)

    (d) Show that in the hyperbolic plane, the length of DA is greater than the length of BC.

18. Consider the set of points and lines indicated below.



    (a) Does this set of points and lines (as drawn) satisfy the incidence axioms?

    (b) Regardless of whether it does satisfy the axioms or not, one can ask if it satisfies the Euclidean parallel postulate, which says that given any line l and point $P \notin l$, there is a unique line m, such that $P \in m$ and $m \| l$. Does this model satisfy the Euclidean parallel postulate?

    (c) Answer the same question for the hyperbolic parallel postulate: given any line l and point $P \not{i}nl$ there are at least two distinct lines m_1 and m_2 such that $P \in m_j$ and $m_j \| l$, for $j = 1, 2$.

19. The defect of a triangle $\triangle ABC$ is

$$\delta ABC := 180^\circ - (\angle A)^\circ - (\angle B)^\circ - (\angle C)^\circ.$$

    (a) Show that the defects of triangles add. In other words, if D is between B and C then

$$\delta ABC = \delta ADB + \delta ADC.$$

    (b) Show that if there exists a rectangle (a quadrilateral with four interior right angles) then there is a right triangle with defect zero. (In fact, the existence of a rectangle implies all right triangles have defect zero, but you needn't prove that.)

    (c) Show that if all right triangles have defect zero then all triangles have defect zero.

20. Given two isometries f and g, we say the conjugate of f by g is $g^{-1} \circ f \circ g$.

    (a) Let r_x be reflection across the x-axis and r_y reflection across the y-axis. Write r_y as the conjugation of r_x by some reflection.

    (b) Show that reflection across any line through the origin is the conjugation of r_x by some reflection. Is this rotation unique?

21. (a) Show that inversion through the unit circle (centered at the origin) is given by

$$z \mapsto \frac{1}{\bar{z}}.$$

    (Argue geometrically.)

    (b) Use the formula above to find a formula for inversion through the circle of center $z_0 = x_0 + iy_0$ and radius r. (Hint: translate the center to the origin, recale, and translate back.)

22. Recall that a Möbius transformation has the form

$$T(z) = \frac{az+b}{cz+d},$$

for some complex numbers a, b, c, d such that $ad - bc \neq 0$.

   (a) Show that, unless T is the identity map, it can have at most 2 fixed points. (Hint: look at the equation $T(z) = z$. You may use the fundamental theorem of algebra, which says that a polynomial of degree n has precisely n roots, counted with multiplicity.)

   (b) Given three distinct points z_0, z_1, z_2, find a Möbius transformation $T(z)$ such that $T(z_0) = 0, T(z_1) = 1, T(z_2) = \infty$.

   (c) Show that, given distinct pairs of points $(z_0, w_0), (z_1, w_1), (z_2, w_2)$, there is a unique Möbius transformation T such that $T(z_0) = w_0, T(z_1) = w_1, T(z_2) = w_2$.

23. Recall that the cross ratio of four complex numbers z_0, z_1, z_2, z_3 is given by

$$(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}.$$

   (a) Show that the cross ratio is invatiant under any Möbius transformation T. In other words, if T is a Möbius transformation then $(z_0, z_1, z_2, z_3) = (T(z_0), T(z_1), T(z_2), T(z_3))$.

   (b) Conclude that a Möbius transformation T which maps the unit disc to itself is an isometry of the Poincaré disc.

   (c) Show that (z_0, z_1, z_2, z_3) is a real number if and only if the four points z_0, z_1, z_2, z_3 all lie on a circle or a line. Conclude that Möbius transformations map circles and lines to circles and lines.

24. Consider the triangle $\triangle ABC$. Let D be the midpoint of AB and E the midpoint of AC.

   (a) Show that in the Eulcidean plane $\triangle ABC$ is similar to $\triangle ADE$.

   (b) Show that in the hyperbolic plane $\triangle ABC$ is not similar to $\triangle ADE$. (Hint: angle sums.)

25. Let T be a Möbius transformation which maps the unit circle to itself, such that $T(1/2) = i/4$. What is $T(2)$? (Hint: you do not need ot compute T.)