Practice Problems

Math 223

April 20, 2005

These problems are in no particular order.

1. Prove that Euclid V is equivalent to the angle sum of any triangle being 180°.

   a) (Euclid V implies the angle sum of a triangle is 180°) Given $\triangle ABC$, let ℓ be the parallel to AB through C. Also let $\angle ECA$ (respectively, $\angle DCB$) be the alternate interior angle to $\angle CAB$ (respectively, $\angle CBA$). Then the sum of the angle measures of $\angle ECA$, $\angle DCB$ and $\angle ACB$ is 180°. By the converse to the alternating interior angle theorem, the same is true for the angle sum of the triangle and we're done with the easy part.

   b) To prove the converse, we need a fact which follows from the addition rule for defects of triangles add (see a later problem);

   Fact: In a neutral geometry, if the angle sum of a triangle is 180°, then the angle sum of any triangle is equal to 180°.

   We next show the converse statement by showing that if triangles have angle sum 180° then Playfair's axiom holds. Let ℓ be a straight line and P a point not on ℓ. We can drop perpendicular PQ to ℓ with Q being the common point of the perpendicular and ℓ. Construct a line m through P parallel to ℓ (say, by constructing congruent alternate interior angles as in the first part). Let n be any other line through P. We need to prove that n meets l. Let X be a point on n on the same side of m as Q. Then ∠QPX is less than 90° . If we can prove the existence of a point S on ℓ which is on the same side of PQ as X and $m\angle QPS > m\angle QPX$, then X is contained in the interior of ∠QPS and Pasch tells us that n has to meet l along the ray from P and going through X. To find S we construct a process that does it. Pick a point S_0 on l on the same side of PQ as X. Extend QS_0 along l so that $|S_0S_1| = |S_0P|$. Then $\triangle PS_0S_1$ is isosceles, and $\angle S_0S_1P = \angle S_0PS_1$. By assumption, there is one triangle with angle sum equal to 180° and, it follows from the Fact stated above that, the angle sum of $\triangle PS_0S_1$ is 180°, thus $m\angle QS_0P = m\angle QS_1P + m\angle S_0PS_1 = 2m\angle QS_1P$, and $m\angle QS_1P = (1/2)m\angle QS_0P$. Repeating this procedure, if necessary, we arrive at a point S_k on ℓ such that $m\angle QS_kP = (1/2k)m\angle QS_0P$. However, $\angle QS_kP + m\angle QPS_k = 90°$. Pick k large enough so that $(1/2k)m\angle QS_0P < 90° - m\angle QPX$, then $\angle QPS_k > m\angle QPX$, We have already shown that our last statement is sufficient to show that n meets l and we are done.

2. Consider the triangle $\triangle ABC$, and let $\vec{BD}$ bisect the angle $\angle ABC$. Show that $\vec{BD} \perp AC$ if and only if $AB \simeq BC$.

   First suppose $\vec{BD} \perp AC$. Then $\triangle ADB \simeq \triangle ADC$ by ASA, so $AB \simeq BC$. Conversely, suppose $AB \simeq AC$. Then $\triangle ADB \simeq \triangle ADC$ by SAS, so $\angle ADB \simeq \angle ADC$. However, these two angles are supplementary, so they can only be congruent if they are both right angles.

3. Recall that a convex polygon P satisfies the property that every line segment joining two points inside P remains inside P. Let P be a convex polygon with vertices $X_1, \dots X_n$ (numbered in order), and let l be a line passing through the side X_1X_2, but not through one of the vertices. Show that l must pass through another side of P. (Hint: use Pasch's axiom for triangles and induction.)

   We will use induction on the number of vertices n of the polygon. Pasch's axiom implies the result is true for $n = 3$. Now we suppose it is true for $n - 1$, and use this to prove it is true for n. Let P be a convex polygon with vertices $X_1, \dots, X_n$, numbered in order as above, and cut P into the convex $n-1$-gon with vertices $X_1, \dots, X_{n-1}$ and the triangle $\triangle X_1X_{n-1}X_n$ by drawing the segment X_1X_{n-1}. This is where we use the convexity of P; if P is not convex, we might get a triangle and the original n-gon. Anyhow, by the induction step, either l passes through X_1X_{n-1}, or it passes through one of the other sides of P'. These other sides of P' are also sides of P, so if l passes through one of these sides we're done. On the other hand, if l passes through X_1X_{n-1}, then it must also pass through either X_1X_n or it must pass through $X_{n-1}X_n$ (by Pasch's axiom), so we're done.

4. Let two circles c_1 and c_2 intersect at points P and Q, and let AP and BP be diameters of c_1 and c_2 (respectively). Show that AB passes through the other intersection point Q.

   Let O_1 be the center of c_1 and O_2 the center of c_2. We wish to show $(\angle PQA)^\circ + (\angle PQB)^\circ = 180^\circ$; if this is true, then A, Q, B are collinear, so Q lies on the line through A and B. Observe that $\angle PQA$ is an inscribed angle with central angle $\angle PO_1A$, and $\angle PQB$ is an inscribed angle with central angle $\angle PO_2B$. Because AP and BP are diameter, we have

$$180^\circ = \frac{360^\circ}{2} = \frac{1}{2}((\angle PO_1A)^\circ + (\angle PO_2B)^\circ) = (\angle PQA)^\circ + (\angle PQB)^\circ.$$

5. Let c_1 and c_2 be mutually tangent circles, meeting at T. Show that the line connecting the centers O_1 and O_2 passes through T.

   Because c_1 and c_2 are mutually tangent, they have a common tangent line, which we call l, at the point T. Pick a point $A \in l$ with $A \neq T$. By the definition of tangent lines, the radial lines $\vec{TO_1}$ and $\vec{TO_2}$ are both perpendicular to l at T. Thus $(\angle ATO_1)^\circ + (\angle ATO_2)^\circ = 90^\circ + 90^\circ = 180^\circ$, and so $\angle ATO_1$ and $\angle ATO_2$ are supplementary. This implies O_1, T, O_2 are collinear.

6. Show that stereographic projection preserves angles. More precisely, if $\Phi : S^2 \backslash \{N\} \to \mathbb{R}^2$ is stereographic projection and c_1 and c_2 circles through N on S^2, meeting at a point P, then the image lines $l_1 = \Phi(c_1)$ and $l_2 = \Phi(c_2)$ meet at the same angle at $Q = \Phi(P)$. (Hint: you don't need to use any formulas.)

   Observe that a circle c on S^2, through N, is the intersection of S^2 with a plane Π through N. These planes will also intersect the horizontal plane of height zero in lines. Consider the circles c_1, c_2 which pass through N, and are cut out by planes Π_1 and Π_2, respectively. The angle between c_1 and c_2 is the same as the angle between Π_1 and Π_2. Moreover, the image lines $l_1 = \Phi(c_1)$ and $l_2 = \Phi(c_2)$ are given by the intersection of Π_1 and Π_2 (respectively) with the height 0 horizontal plane. Thus the angle between l_1 and l_2 is also the angle between Π_1 and Π_2, which completes the proof.

7. Prove the law of cosines:

   $$c^2 = a^2 + b^2 - 2ab\cos(\theta),$$

   where a, b, c are side lengths of a triangle and θ is the angle opposite c.

   We call the vertices of this triangle A, B, C, where A is opposite the side of length a, B is opposite the side of length b, and C is opposite the side of length c. If $(\theta)^\circ = 90^\circ$, then this is the Pythagorean theorem. In any case, drop a perpendicular from A to the line through B and C, which intersects the line at the point D. There are three cases to consider: D can be between B and C, B can be between C and D, or C can be between B and D. (At this point, it might help you to draw a picture.) In all three cases, we have

   $$|AD|^2 = b^2 - |CD|^2 = c^2 - |BD|^2.$$

   If D is between B and C, then we also have

   $$a = |BD| + |CD|,$$

   so we can write $a^2 = (|BD| + |CD|)^2 = |BD|^2 + |CD|^2 + 2|BD||CD|$. Now solve for $|BD|^2$:

   $$|BD|^2 = a^2 - |CD|^2 - 2|BD||CD| = c^2 - b^2 + |CD|^2,$$

   which we can rearrange to read

   $$c^2 = a^2 + b^2 - 2|CD|^2 - 2|BD||CD| = a^2 + b^2 - 2|CD|(|CD| + |BD|) = a^2 + b^2 - 2a|CD|.$$

   Finally, we observe that $|CD| = b\cos\theta$, to get

   $$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

   If B is between C and D, we have

   $$|CD| = a + |BD|,$$

   so we can write $|CD|^2 = a^2 + |BD|^2 + 2a|BD|$. This is also equal to $b^2 - c^2 + |BD|^2$. Thus we have

   $$c^2 = b^2 - a^2 + 2a|BD|.$$

   However, $b\cos\theta = -(|BD| + a)$, so we get

   $$c^2 = b^2 - a^2 + 2a(a - b\cos\theta) = a^2 + b^2 - 2ab\cos\theta.$$

   The last case is similar (reverse the roles of B and C).

8. Let f and g be isometries. Show that $f \circ g$ is also an isometry.

   Start with two points $P, Q \in \mathbb{R}^2$. Then

   $$\text{dist}(f \circ g(P), f \circ g(Q)) = \text{dist}(f(g(P)), f(g(Q))) = \text{dist}(g(P), g(Q)) = \text{dist}(P, Q),$$

   so $f \circ g$ is an isometry.

9. Recall that a regular n-gon is an n-sided polygon such that all its sides and interior angles are congruent. Show that a regular n-gon has n lines of reflection symmetry. (Hint: look at the bisectors of the interior angles and the perpendicular bisectors of the sides.)

   Let's use complex numbers for the solution:

   We can scale down the regular n-gon and move it so that it is centered at 0. Then the vertices all lie on the unit circle and are equi-spaced there. We can assume that 1 is a vertex, then so is $e^{2\pi ik/n}$ for $k = 0, 1, \ldots, n-1$. If n is even, then -1 is also a vertex and we see that the lines connecting opposite vertices are lines of symmetry. Otherwise, $(n-1)/2$ of the vertices lie above the x-axis and the same number lie below. The midpoint of the side opposite 1 lies on the x-axis and the x-axis is a line of symmetry of the n-gon, hence so is every line connecting a vertex to the midpoint of the opposite side.

10. Let r_l be reflection through the line l and r_m reflection through the line m. Show that $r_m \circ r_l \circ r_m = r_{l'}$, where l' is the reflection of l through m. (Hint: let A and B be distinct points on l. Show that $r_m(A)$ and $r_m(B)$ are fixed points of $r_m \circ r_l \circ r_m$.)

    Following the hint, we get ℓ' is the line through $r_m(A)$ and $r_m(B)$. ℓ' must be fixed by $S := r_m \circ r_\ell \circ r_m$. If S fixes a point off ℓ', it is the identity, otherwise S is the reflection in ℓ'. If S is the identity, then r_ℓ is the identity since $r_m \circ S \circ r_m = r_\ell = r_m \circ \text{identity} \circ r_m = \text{identity}$.

11. Let r_m be reflection through the line m. Show that r_m maps the line l to itself (i.e. l is invariant under r_m) if and only if $l \perp m$.

    If $\ell = m$, it is invariant under r_m. Otherwise choose any point P on ℓ which is not on m. It is mapped to the point which is both equidistant from m and on the perpendicular to m through P. Since the segment from P to $r_m(P)$ lie both on ℓ and the perpendicular, ℓ is the perpendicular.

12. Let Γ be the lattice consisting of the points

    $$(n, m), m \text{ even}, \qquad (n + \frac{1}{2}, m), m \text{ odd}.$$

    Find two non-parallel translations which preserve Γ.

    The fairly obvious one is translation to the right (or left) by one unit. The less obvious one is translation up by one unit and right by half a unit. How do you find these translations? Consider four neighboring lattice point $(n, m), (n+1, m), (n+1/2, m+1), (n+3/2, m+1)$, where m is any integer and m is an even integer. These lattice points are the corners of a parallelogram (minimal in the sense that no lattice points lie inside the parallelogram). The sides of this parallelogram give you displacement vectors for the translation.

13. Let $T = r_2 \circ r_1$ be a translation, where r_1 and r_2 are reflections. Find a formula for T^{-1} in terms of r_1 and r_2.

    The two salient points to recall is that (i) inversion reverses the order of operations and (ii) any reflection is its own inverse. Thus

    $$T^{-1} = (r_2 \circ r_1)^{-1} = r_1^{-1} \circ r_2^{-1} = r_1 \circ r_2.$$

14. Let T be a translation which is not the identity. Show that if l_1 and l_2 are two lines which are invariant under T, then $l_1 \| l_2$. (Recall that a line l is invariant under T if and only is T maps l to itself.)

    Let A lie on ℓ_1 and B on ℓ_2 be the point where the perpendicular to ℓ_2 through A meets ℓ_2. Since T is a translation all points are moved the same distance, $|AT(A)| = |BT(B)|$. Since T is a motion, $|AB| = |T(A)T(B)|$. So $\square ABCD$ has opposite sides of equal length. Therefore it is a parallelogram. Two of the sides are segments in ℓ_1 and ℓ_2.

15. Let R be a rotation with fixed point O, which is not the identity. Show that if l is a line which does not pass through O, then R does not map l to itself.

    Using complex numbers, $R(z) = e^{i\theta} z$ with θ not a multiple of 2π. Unless $\theta = \pi$ (or more precisely, an odd multiple of π) or $z = 0$, $R(z), R \circ R(z)$ and $R \circ R \circ R(z)$ are not collinear. So no line through 0 can be invariant unless $\theta = \pi$ or R is a rotation by 180°.

16. Let R be a rotation. Show that if R has a fixed line then R is rotation by 180°.

    This was proved in the previous problem if the center of rotation is 0. There are two things you can do now. Relabel the plane so that the center of rotation is 0. Alternatively, let T be the translation taking 0 to the center of rotation. Then $R_1 := T \circ R \circ T^{-1}$ is a rotation fixing 0 and takes an invariant line for R to an invariant line for R_1. Now apply the previous argument.

17. A Saccheri quadrilateral $\square ABCD$ satisfies $AB \simeq CD$ and $(\angle ABC)^\circ = 90^\circ = (\angle BCD)^\circ$. Also, let M be the midpoint of BC and M' the midpoint of DA.

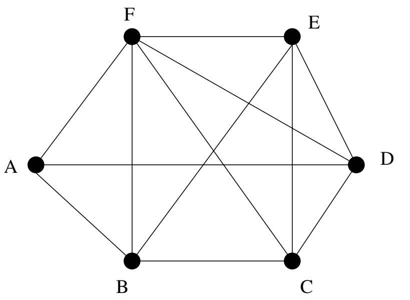
    (a) Show that in both the Euclidean and hyperbolic plane, $\angle BAD \simeq \angle CDA$. (Hint: congruent triangles.)
    First observe that $\triangle ABC \simeq \triangle DCB$ by SAS. This implies $AC \simeq DB$, $\angle ACB \simeq \angle DBC$, and $\angle BAC \simeq \angle CDB$. Because $(\angle DBC)^\circ + (\angle DBA)^\circ = 90^\circ = (\angle ACB)^\circ + (\angle ACD)^\circ$, we have $\angle DBA \simeq \angle ACD$. Then $\triangle ABD \simeq \triangle DCA$ by SAS, so $\angle BAD \simeq \angle CDA$.

    (b) Show that in both the Euclidean and hyperbolic plane, MM' meets both the sides BC and DA at right angles. (Hint: more congruent triangles.)
    We have $\triangle BAM' \simeq \triangle DCM'$ by SAS, so $BM' \simeq CM'$. Then $\triangle BMM' \simeq \triangle CMM'$ by SSS, so $\angle BMM' \simeq \angle CMM'$. However, these are supplementary angles, so they can only be congruent if they are both right angles. Similarly, $\triangle AMM' \simeq \triangle DMM'$ by SSS, so $\angle AM'M \simeq \angle DM'M$, which implies they are both right angles.

    (c) Show that in the hyperbolic plane, $(\angle BAD)^\circ < 90^\circ$. (Hint: angle sum of a triangle.)
    Recall that the angle sum of any triangle in the hyperbolic plane is strictly less than 180°. This implies the angle sum of any convex quadilateral is strictly less than 360° (split the quadrilateral using either diagonal). The quadrilateral $\square ABMM'$ already has three right angles: $\angle ABM, \angle BMM', \angle MM'A$. Thus $(\angle BAD)^\circ = (\angle BAM')^\circ < 90^\circ$, in order that the angle sum of $\square ABMM'$ is less than 360°. A similar argument shows $(\angle CDA)^\circ < 90^\circ$.

    (d) Show that in the hyperbolic plane, the length of DA is greater than the length of BC.

18. Consider the set of points and lines indicated below.



    (a) Does this set of points and lines (as drawn) satisfy the incidence axioms?
    No. There are not lines connecting all possible pairs of distinct points (C and A, for instance).

    (b) Regardless of whether it does satisfy the axioms or not, one can ask if it satisfies the Euclidean parallel postulate, which says that given any line l and point $P \notin l$, there is a unique line m, such that $P \in m$ and $m \| l$. Does this model satisfy the Euclidean parallel postulate?
    No. Consider the line AB and the point E. There are three lines through E which are parallel to AB; they're EF, ED, EC.

    (c) Answer the same question for the hyperbolic parallel postulate: given any line l and point $P \not{i}nl$ there are at least two distinct lines m_1 and m_2 such that $P \in m_j$ and $m_j \| l$, for $j = 1, 2$.
    You can check that this model does satisfy the hyperbolic parallel postulate. For instance, the line through A which are parallel to BE are AF, AD.

19. The defect of a triangle $\triangle ABC$ is

$$\delta ABC := 180^\circ - (\angle A)^\circ - (\angle B)^\circ - (\angle C)^\circ.$$

    (a) Show that the defects of triangles add. In other words, if D is between B and C then

$$\delta ABC = \delta ADB + \delta ADC.$$

    Observe that $(\angle BDA)^\circ + (\angle CDA)^\circ = 180^\circ$, because these two angles are supplementary. We also have $(\angle BAC)^\circ = (\angle DAB)^\circ + (\angle DAC)^\circ$. So

$$\begin{aligned} \delta ADB + \delta ADC &= 180^\circ - (\angle ABC)^\circ - (\angle BDA)^\circ - (\angle DAB)^\circ + 180^\circ - (\angle ACB)^\circ - (\angle CDA)^\circ - (\angle DAC)^\circ \\ &= 360^\circ - (\angle BDA)^\circ - (\angle CDA)^\circ - (\angle ABC)^\circ - (\angle ACB)^\circ - (\angle BAC)^\circ \\ &= 180^\circ - (\angle ABC)^\circ - (\angle ACB)^\circ - (\angle BAC)^\circ = \delta ABC. \end{aligned}$$

(b) Show that if there exists a rectangle (a quadrilateral with four interior right angles) then there is a right triangle with defect zero. (In fact, the existence of a rectangle implies all right triangles have defect zero, but you needn't prove that.)

Suppose there exists a rectangle $\square ABCD$, with AB and CD being opposite sides. Then $\triangle ABC$ and $\triangle ACD$ are two right triangles (with right angles at vertices B and D, respectively). Moreover, the angle sum of $\square ABCD$ must equal the angle sum of $\triangle ABC$ plus the angle sum of $\triangle ACD$. This sum of angle sums is 360°, so each right triangle $\triangle ABC$ and $\triangle ACD$ must have angle sum 180°. In other words, these two right triangles have defect zero.

(c) Show that if all right triangles have defect zero then all triangles have defect zero.

Let $\triangle ABC$ be a triangle without a right angle, and drop a perpendicular from A to the line through B and C. Call the point where this perpendicular intersects the line D. There are three cases: D is between B and C, B is between C and D, and C is between B and D. In the first case, we have split $\triangle ABC$ into two right triangles, $\triangle ABD$ and $\triangle ACD$, both of which have defect zero. Then $\delta ABC = \delta ABD + \delta ACD = 0$. In the case where B is between C and D, we have split $\triangle ACD$ into $\triangle ABC$ and $\triangle ABD$. However, both $\triangle ACD$ and $\triangle ABD$ are right triangles, so $0 = \delta ACD = \delta ABC + \delta ABD = \delta ABC$. The last case is very similar.

20. Given two isometries f and g, we say the conjugate of f by g is $g^{-1} \circ f \circ g$.

(a) Let r_x be reflection across the x-axis and r_y reflection across the y-axis. Write r_y as the conjugation of r_x by some reflection.

Let R be rotation in the counter-clockwise direction by 90°; then $r_y = R r_x R^{-1}$, which we can verify by checking three non-collinear points. Let's pick the points $(0,0), (1,0), (0,1)$. All three maps r_x, r_y, R fix $(0,0)$, so there's nothing to check with that point. Now we compute:

$$R(r_x(R^{-1}(1,0))) = R(r_x(0,-1)) = R(0,1) = (-1,0) = r_y(1,0).$$

Similarly,

$$R(r_x(R^{-1}(0,1))) = R(r_x(1,0)) = R(1,0) = (0,1) = r_y(0,1).$$

Since the two isometries $R \circ r_x \circ R^{-1}$ and r_y agree on three non-collinear points, they must be the same.

(b) Show that reflection across any line through the origin is the conjugation of r_x by some reflection. Is this rotation unique?

Let l be a line through the origin, which make an angle θ with the x-axis, and let R be rotation in the counter-clockwise direction by the angle θ. Then $r_l = R \circ r_x \circ R^{-1}$, which we can verify in the same way we did for the previous part. First observe that all isometries fix the origin, so $r_l(0,0) = (0,0) = R \circ r_x \circ R^{-1}(0,0)$. Next pick a point $P \in l$, so that $r_l(P) = P$. Also, $R^{-1}(P)$ lies on the x-axis, so $r_x(R^{-1}(P)) = R^{-1}(P)$. Thus $R(r_x(R^{-1}(P))) = R(R^{-1}(P)) = P$. Finally, we choose Q such that the ray $\vec{OQ}$ is perpendicular to l. Then $r_l(Q) = -Q$. Also, $r_x(R^{-1}(Q)) = -R^{-1}(Q)$. Thus $R(r_x(R^{-1}(Q))) = R(-R^{-1}(Q)) = -Q = r_l(Q)$. Since the two isometries $R \circ r_x \circ R^{-1}$ and r_y agree on three non-collinear points, they must be the same.

21. (a) Show that inversion through the unit circle (centered at the origin) is given by

$$z \mapsto \frac{1}{\bar{z}}.$$

(Argue geometrically.)

Recall that the complex conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$. First observe that

$$|z| \cdot \frac{1}{|\bar{z}|} = 1,$$

so $1/\bar{z}$ has the right magnitude. Next observe

$$\frac{1}{\bar{z}} = \frac{z}{z\bar{z}} = \frac{z}{|z|^2},$$

so $1/\bar{z}$ lies on the ray starting at 0 and passing through z, which means $1/\bar{z}$ points in the right direction. Thus $1/\bar{z}$ must be the inverse of z through the unit circle.

(b) Use the formula above to find a formula for inversion through the circle of center $z_0 = x_0 + iy_0$ and radius r. (Hint: translate the center to the origin, recale, and translate back.)

Let γ be a circle of center z_0 and radius R. If we translate by $-z_0$, we get a circle centered at the origin, and if we then rescale by $1/R$ we get the unit circle. So the transformation

$$w = \frac{1}{R}(z - z_0)$$

maps γ to the unit circle. The inverse point to w is

$$\frac{1}{\bar{w}} = \frac{R}{\bar{z} - \bar{z}_0}.$$

We must then transform back to the original circle, first rescaling by R and then translating by z_0. Then we get the inverse point

$$\frac{R}{\bar{W}} + z_0 = \frac{R^2}{\bar{z} - \bar{z}_0} + z_0.$$

22. Recall that a Möbius transformation has the form

$$T(z) = \frac{az + b}{cz + d},$$

for some complex numbers a, b, c, d such that $ad - bc \neq 0$.

(a) Show that, unless T is the identity map, it can have at most 2 fixed points. (Hint: look at the equation $T(z) = z$. You may use the fundamental theorem of algebra, which says that a polynomial of degree n has precisely n roots, counted with multiplicity.)

Any fixed points z satisfy the equation

$$z = T(z) = \frac{az + b}{cz + d} \Leftrightarrow cz^2 + (d - a)z - b = 0.$$

This is a quadratic equation in z, so it has precisely 2 solutions (counted with multiplicity.

(b) Given three distinct points z_0, z_1, z_2, find a Möbius transformation $T(z)$ such that $T(z_0) = 0, T(z_1) = 1, T(z_2) = \infty$.

We need $T(z_0) = 0$, so T should have a factor of $(z - z_0)$. Also, $T(z_2) = \infty$, so T must also have a factor of $1/(z - z_2)$. So far we have

$$T(z) = \lambda \frac{z - z_0}{z - z_2},$$

for some constant λ which we can choose. Finally, we use $T(z_1) = 1$ to get $\lambda = (z_1 - z_2)/(z_1 - z_0)$, so

$$T = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)}.$$

(c) Show that, given distinct pairs of points $(z_0, w_0), (z_1, w_1), (z_2, w_2)$, there is a unique Möbius transformation T such that $T(z_0) = w_0, T(z_1) = w_1, T(z_2) = w_2$.

Let T_1 be the Möbius transformation such that $T_1(z_0) = 0, T_1(z_1) = 1, T_1(z_2) = \infty$. Also let T_2 be the Möbius transformation such that $T_2(w_0) = 0, T_1(w_1) = 1, T_1(w_2) = \infty$, and define

$$T(z) = T_2^{-1}(T_1(z)).$$

Then $T(z_0) = T_2^{-1}(T_1(z_0)) = T_2^{-1}(0) = w_0$. Similarly, $T(z_1) = w_1$ and $T(z_2) = w_2$.

How about uniqueness? Suppose $\hat{T}$ were another Möbius transformation such that $\hat{T}(z_0) = w_0, \hat{T}(z_1) = w_1, \hat{T}(z_2) = w_2$. Then $\hat{T}^{-1} \circ T$ fixes three points, so it must be the identity map. So in this case,

$$\hat{T}^{-1} \circ T = \mathrm{Id} \Leftrightarrow T = \hat{T}.$$

23. Recall that the cross ratio of four complex numbers z_0, z_1, z_2, z_3 is given by

$$(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}.$$

(a) Show that the cross ratio is invatiant under any Möbius transformation T. In other words, if T is a Möbius transformation then $(z_0, z_1, z_2, z_3) = (T(z_0), T(z_1), T(z_2), T(z_3))$.
Define

$$\hat{T}(z) = (z, z_1, z_2, z_3) = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}, \qquad \check{T}(z) = (z, T(z_1), T(z_2), T(z_3)) = \frac{(z - T(z_2))(T(z_1) - T(z_3))}{(z - T(z_3))(T(z_1) - T(z_2))}.$$

Then $\hat{T} \circ T^{-1}$ maps $T(z_1)$ to 1, $T(z_2)$ to 0, and $T(z_3)$ to ∞, and so does $\check{T}$. Thus we must have $\hat{T} \circ T^{-1} = \check{T}$. This implies

$$(z, z_1, z_2, z_3) = \hat{T}(z)) = \hat{T}(T^{-1}(T(z))) = \check{T}(T(z)) = (T(z), T(z_1), T(z_2), T(z_3)).$$

Evaluating this last expression at $z = z_0$ proves the result.

(b) Conclude that a Möbius transformation T which maps the unit disc to itself is an isometry of the Poincaré disc.
Recall that the distance function in the Poincaré disc is the logarithm of a cross ration. Because Möbius transformations preserve the cross ratio, they also preserve this logarithm. Thus a Möbius transformation mapping the unit disc to itself preserve distance, which means it's an isometry of the Poincaré disc.

(c) Show that (z_0, z_1, z_2, z_3) is a real number if and only if the four points z_0, z_1, z_2, z_3 all lie on a circle or a line. Conclude that Möbius transformations map circles and lines to circles and lines.
We define the Möbius transformation

$$T(z) = (z, z_1, z_2, z_3) = \frac{az + b}{cz + d}.$$

The cross ratio is real precisely when

$$T(z) = T\bar(z) \Leftrightarrow \frac{az + b}{cz + d} = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} \Leftrightarrow 0 = (a\bar{c} - \bar{a}c)|z|^2 + (a\bar{d} - \bar{b}c)z + (b\bar{c} - \bar{a}d)\bar{z} + (b\bar{d} - \bar{b}d).$$

If $a\bar{c} - \bar{a}c = 0$, then we let $\alpha = a\bar{d} - b\bar{c}$ and $\beta = b\bar{d} - \bar{b}d$, and rewrite the last equation as

$$\Im(\alpha z + \beta) = 0,$$

which is the equation of a line. In the other case, we divide through by $a\bar{c} - \bar{a}c$. Letting

$$\gamma = \frac{a\bar{d} - \bar{b}c}{a\bar{c} - \bar{a}c}, \qquad \delta = \frac{b\bar{d} - \bar{b}d}{a\bar{c} - \bar{a}c},$$

we can rewrite the equation as

$$|z|^2 + \gamma z + \bar{\gamma}z + \delta = 0 \Leftrightarrow |z + \bar{\gamma}|^2 = -\delta + |\gamma|^2.$$

This is the equation of a circle centered at $\bar{\gamma}$.
The fact that Möbius transformations preserve the cross ratio then implies that Möbius transformations send circles and lines to circles and lines.

24. Consider the triangle $\triangle ABC$. Let D be the midpoint of AB and E the midpoint of AC.

(a) Show that in the Euclidean plane $\triangle ABC$ is similar to $\triangle ADE$.
Note that the two triangles share the angle $\angle BAC$. Also, because D and E are midpoints, we have $|AB| = 2|AD$ and $|AC| = 2|AE|$. Thus by SAS similarity, the two triangles are similar.

(b) Show that in the hyperbolic plane $\triangle ABC$ is not similar to $\triangle ADE$. (Hint: angle sums.)
Two triangles in the hyperbolic plane are similar if and only if they're congruent. In this case, $\triangle ADE$ and $\triangle ABC$ can't be congruent because their corresponding sides are not congruent.
We can also see they're not similar directly. Observe that $(\angle AED)^\circ + (\angle DEC)^\circ = 180^\circ = (\angle ADE)^\circ + (\angle EDB)^\circ$. Because we're in the hyperbolic plane,

$$360^\circ > (\angle DEC)^\circ + (\angle EDB)^\circ + (\angle DBC)^\circ + (\angle ECB)^\circ = 180^\circ - (\angle AED)^\circ + 180^\circ - (\angle ADE)^\circ + (\angle ABC)^\circ + (\angle ACB)^\circ,$$

which we can rearange to read $(\angle AED)^\circ + (\angle ADE)^\circ > (\angle ABC)^\circ + (\angle ACB)^\circ$. If the two triangles we similar, then the angle sums would have to be the same.

25. Let T be a Möbius transformation which maps the unit circle to itself, such that $T(1/2) = 2i$. What is $T(i/4)$? (Hint: you do not need to compute T.)

Recall that Möbius transformations send inverse points to inverse points. Note that $1/2$ and 2 are inverses of each other, with repsect to the unit circle, and T sends the unit circle to itself. Thus $T(2)$ is the inverse of $T(1/2)$ with respect to the unit circle, which is $4i$.