

Practice Problems

Math 223

March 17, 2005

These problems are in no particular order.

1. Let $\angle APB$ be an inscribed angle in a circle γ . Prove that $(\angle APB)^\circ = 90^\circ$ if and only if the chord AB is a diameter of γ .
2. Consider a circle γ and a point P outside γ , and let l_1, l_2 be the two lines tangent to γ passing through P . Also let O be the center of γ , let l_1 contact γ at A , and let l_2 contact γ at B . Prove that \vec{PO} bisects $\angle APB$. (It might help to draw a picture.)
3. (a) Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ preserve distances. In other words, Φ satisfies

$$\|\Phi(X) - \Phi(Y)\| = \|X - Y\|.$$

Show that Φ also preserves angles. In other words, If two lines l_1 and l_2 meet at an angle θ , then their images $\Phi(l_1)$ and $\Phi(l_2)$ also meet at an angle θ . (Hint: congruent triangles.)

- (b) Does your argument work of $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, or is it particular to two dimensions? Be sure to explain your answer.
4. Find the area and perimeter of the inscribed regular heptagon (7-gon) in a unit circle. (Hint: use trigonometric functions. You may leave your answer in terms of $\sin(2\pi/7)$ and $\cos(2\pi/7)$.)
5. (a) Fix distinct points P_1, P_2 in \mathbb{R}^2 and positive numbers r_1, r_2 . Consider $U_1 = \{Q \in \mathbb{R}^2 \mid \text{dist}(P_1, Q) = r_1\}$ and $U_2 = \{Q \in \mathbb{R}^2 \mid \text{dist}(P_2, Q) = r_2\}$. How many points can $U_1 \cap U_2$ contain? List all the possibilities.
(b) Under what conditions is $U_1 \cap U_2$ nonempty?
(c) Under what conditions does $U_1 \cap U_2$ contain precisely one point?
(d) Suppose $U_1 \cap U_2$ contains more than one point. Let $P_3 \in \mathbb{R}^2$ be distinct from P_1 and P_2 , and let $Q \in U_1 \cap U_2$. Under what conditions does the circle with center P_3 and radius $\text{dist}(P_3, Q)$ contain only one of the points in $U_1 \cap U_2$?
(e) Given three points P_1, P_2, P_3 in the plane and another point Q , what conditions on P_1, P_2, P_3 so that Q is completely determined by $\text{dist}(Q, P_1), \text{dist}(Q, P_2), \text{dist}(Q, P_3)$?
6. Prove that Playfair's axiom follows from Euclid I–IV and the following condition: given two lines l and m there are two distinct points $P, Q \in m$ such that the distance from P to l is equal to the distance from Q to l .
7. Recall that a polygon is a shape in the plane whose boundary is a union of line segments. Moreover, a polygon is convex if any segment joining points on its boundary lies inside the polygon.
 - (a) Let D be convex polygon with vertices P_1, \dots, P_n , where P_i and P_{i+1} are adjacent vertices. Prove that one can decompose D into $n - 2$ triangles. (Hint: induction.)
 - (b) Let $\triangle ABC$ and $\triangle DEF$ be similar triangles with $|AB| = k|DE|$. Prove that $\text{area}(\triangle ABC) = k^2 \text{area}(\triangle DEF)$.
 - (c) One can make sense of similar polygons in the same way one makes sense of similar triangles: two polygons are similar if their corresponding angles are congruent and their corresponding sides are proportionate. Let D and D' be similar convex polygons, where D has vertices P_1, \dots, P_n and D' has vertices P'_1, \dots, P'_n . Also suppose $|P_1P_2| = k|P'_1P'_2|$. Prove that $\text{area}(D) = k^2 \text{area}(D')$.