Solutions to the Practice Problems

Math 223

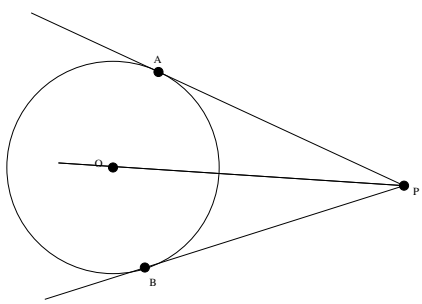
March 21, 2005

1. Let $\angle APB$ be an inscribed angle in a circle γ. Prove that $(\angle APB)^\circ = 90^\circ$ if and only if the chord AB is a diameter of γ.

   Let O be the center of γ. First suppose that AB is a diameter. Then $180^\circ = (\angle AOB)^\circ = 2(\angle APB)^\circ$, and so $\angle APB$ is a right angle. Next suppose that $(\angle APB)^\circ = 90^\circ$. Then $(\angle AOB) = 2(\angle APB)^\circ = 180^\circ$, and so the points A, O, and P are collinear. Thus $|AB| = |AO| + |OB|$, which is twice the radius of γ, and so AB is a diameter.

2. Consider a circle γ and a point P outside γ, and let l_1, l_2 be the two lines tangent to γ passing through P. Also let O be the center of γ, let l_1 contact γ at A, and let l_2 contact γ at B. Prove that $\vec{PO}$ bisects $\angle APB$. (It might help to draw a picture.)

   Here's the picture you might have drawn:



   We know that $\angle OAP$ and $\angle OBP$ are both right angles, because l_1 and l_2 are both tangent to γ. Also, $OA \simeq OB$ and $OP \simeq OP$, and so $\triangle OAP \simeq \triangle OBP$ by hypotenuse-leg for right triangles. Thus $\angle APO \simeq \angle BPO$, which is what we wanted to show.

3. (a) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ preserve distances. In other words, Φ satisfies

   $$\|\Phi(X) - \Phi(Y)\| = \|X - Y\|.$$

   Show that Φ also preserves angles. In other words, If two lines l_1 and l_2 meet at an angle θ, then their images $\Phi(l_1)$ and $\Phi(l_2)$ also meet at an angle θ. (Hint: congruent triangles.)

   Let l_1 and l_2 be lines which meet at P, forming an angle θ. Also choose $P_1 \in l_1$ and $P_2 \in l_2$ such that P_1, P_2, P are distinct (so $\theta = \angle P_1PP_2$). Then

   $$\|P_1 - P\| = \|\Phi(P_1) - \Phi(P)\|, \qquad \|P_2 - P\| = \|\Phi(P_2) - \Phi(P)\|, \qquad \|P_1 - P_2\| = \|\Phi(P_1) - \Phi(P_2)\|,$$

   so the triangles $\triangle P_1P_2P$ and $\triangle \Phi(P_1)\Phi(P_2)\Phi(P)$ are congruent by SSS. Thus $\angle P_1PP_2 \simeq \angle \Phi(P_1)\Phi(P)\Phi(P_2)$, which is what we wanted to show.

   (b) Does your argument work of $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$, or is it particular to two dimensions? Be sure to explain your answer.

   This proof still works in three dimensions (in fact, in any dimension), because SSS still holds as a congruence rule for triangles.

4. Find the area and perimeter of the incribed regular heptagon (7-gon) in a unit circle. (Hint: use trigonometric functions. You may leave your answer in terms of $\sin(\pi/7)$ and $\cos(\pi/7)$.)

   Label the center of the circle O and the vertices of the regular heptagon as $P_1, P_2, \ldots, P_7$ (with P_i adjacent to P_{i+1}). Then the triangles $\triangle P_iOP_{i+1}$ divide the heptagon into seven congruent triangles (congruent by SSS). Also, $|OP_i| = 1$ and $\angle P_iPP_{i+1} = 2\pi/7$. We want to find P_iP_{i+1}, and the heights of these triangles. Let M_i be the midpoint of P_iP_{i+1}. Then $\triangle P_iOM_i \simeq \triangle P_{i+1}OM_i$ by SSS, so $\angle P_iM_iO$ and $\angle P_{i+1}M_iO$ are both right angles (because they're congruent supplementary angles). This means the triangle $\triangle P_iOP_{i+1}$ has height $\cos(\pi/7)$ and base $2\sin(\pi/7)$. Thus the perimeter of the heptagon is $14\sin(\pi/7) \approx 6.074$ and the area is $7\cos(\pi/7)\sin(\pi/7) \approx 2.736$

5. (a) Fix distinct points P_1, P_2 in $\mathbb{R}^2$ and positive numbers r_1, r_2. Consider $U_1 = \{Q \in \mathbb{R}^2 \mid \text{dist}(P_1, Q) = r_1\}$ and $U_2 = \{Q \in \mathbb{R}^2 \mid \text{dist}(P_2, Q) = r_2\}$. How many points can $U_1 \cap U_2$ contain? List all the possibilities.

We may as well choose Cartesian coordinates so that $P_1 = (0, 0)$ and $P_2 = (\bar{x}, 0)$, which amounts to doing a translation and a rotation. Then U_1 is given by $x^2 + y^2 = r_1^2$ and U_2 is given by $(x - \bar{x})^2 + y^2 = r_2^2$. The intersection $U_1 \cap U_2$ is the set of simultaneous solutions to these equations, which we write as

$$r_1^2 - x^2 = r_2^2 - (x - \bar{x})^2 \Leftrightarrow x = \frac{\bar{x}^2 + r_1^2 - r_2^2}{2\bar{x}}.$$

Knowing x, we can plug back into either quadratic ($x^2 + y^2 = r_1^2$ or $(x - \bar{x})^2 + y^2 = r_2^2$) to find at most two values of y. One solution and no solutions are also both possibilities.

Observe that in the case where $U_1 \cap U_2$ contains two points, they have the form $(x_*, \pm y_*)$.

(b) Under what conditions is $U_1 \cap U_2$ nonempty?

We look at the quadratics for y:

$$y^2 = r_1^2 - \frac{(\bar{x}^2 + r_1^2 - r_2^2)^2}{4\bar{x}^2}, \qquad y^2 = r_2^2 - [\frac{(\bar{x}^2 + r_1^2 - r_2^2 - 2\bar{x}^2)^2}{4\bar{x}^2}$$

We need both expressions for y^2 to be non-negative, which happens if and only if $\bar{x} - r_2 \leq r_1$ and $-r_1 \leq \bar{x} + r_2$.

(c) Under what conditions does $U_1 \cap U_2$ contain precisely one point?

This happens if and only if we have equality in one of the equations above: either $r_1 = \bar{x} - r_2$ or $-r_1 = \bar{x} + r_2$.

(d) Suppose $U_1 \cap U_2$ contains more than one point. Let $P_3 \in \mathbb{R}^2$ be distinct from P_1 and P_2, and let $Q \in U_1 \cap U_2$. Under what conditions does the circle with center P_3 and radius $\text{dist}(P_3, Q)$ contain only one of the points in $U_1 \cap U_2$?

Recall that in this case, $U_1 \cap U_2 = \{(x_*, \pm y_*)\}$. Let's suppose that the circle c with center P_3 and radius $\text{dist}(P_3, Q)$ contains both of the points in $U_1 \cap U_2$. Giving P_3 the coordinates (x_3, y_3), we see

$$(x_3 - x_*)^2 + (y_3 - y_*)^2 = (x_3 - x_*)^2 + (y_3 + y_*)^2,$$

which implies $y_3 = 0$. Conversely, if $y_3 \neq 0$ then the circle c will contain only one of $(x_*, \pm y_*)$. Geometrically, this is the same thing as saying P_1, P_2, P_3 are not collinear.

(e) Given three points P_1, P_2, P_3 in the plane and another point Q, what conditions on P_1, P_2, P_3 so that Q is completely determined by $\text{dist}(Q, P_1)$, $\text{dist}(Q, P_2)$, $\text{dist}(Q, P_3)$?

Now we only need to interpret what we've already done. The last part says it suffices to have three non-collinear points P_1, P_2, P_3, so that any point Q is uniquely determined by the three distances $\text{dist}(Q, P_1)$, $\text{dist}(Q, P_2)$, $\text{dist}(Q, P_3)$.

6. Prove that Playfair's axiom follows from Euclid I–IV and the following condition: given two lines l and m there are two distinct points $P, Q \in m$ such that the distance from P to l is equal to the distance from Q to l.

7. Recall that a polygon is a shape in the plane whose boundary is a union of line segments. Moreover, a polygon is convex if any segment joining points on its boundary lies inside the polygon.

(a) Let D be convex polygon with vertices $P_1, \ldots, P_n$, where P_i and P_{i+1} are adjacent vertices. Prove that one can decompose D into $n - 2$ triangles. (Hint: induction.)

Observe that a 3-gon is a triangle, so the result is true for $n = 3$. Suppose the result is true for n, and let D be a convex $(n + 1)$–gon with vertices $P_1, \ldots P_{n+1}$. Then draw the segment $P_n P_1$, which decomposes D into the triangle $\triangle P_n P_{n+1} P_1$, and the convex n-gon D', which has vertices $P_1, \ldots, P_n$. By induction, D' decomposes into $n - 2$ triangles, so D decomposes into $n - 1 = (n + 1) - 2$ triangles. This completes the proof.

(b) Let $\triangle ABC$ and $\triangle DEF$ be similar triangles with $|AB| = k|DE|$. Prove that $\text{area}(\triangle ABC) = k^2 \text{area}(\triangle DEF)$.

Observe that the bases b_1 and b_2 of $\triangle ABC$ and $\triangle DEF$ (respectively) are related by $b_1 = kb_2$. Similarly, the heights h_1 and h_2 are releated by $h_1 = kh_2$. Thus the areas are related by

$$\text{area}(\triangle ABC) = b_1 h_1 = k^2 b_2 h_2 = k^2 \text{area}(\triangle DEF).$$

(c) One can make sense of similar polygons in the same way one makes sense of similar triangles: two polygons are similar if their corresponding angles are congruent and their corresponding sides are proportionate. Let D and D' be similar convex polygons, where D has vertices $P_1, \ldots, P_n$ and D' has vertices $P'_1, \ldots, P'_n$. Also suppose $|P_1P_2| = k|P'_1P'_2|$. Prove that $\text{area}(D) = k^2\text{area}(D')$.

Decompose D and D' into $n-2$ triangles, by connecting corresponding vertices (so that the individual triangles are also similar). To establish notation, we label the triangles as $T_1, \ldots, T_{n-2}$ in D and $T'_1, \ldots, T'_{n-2}$, where T_i and T'_i are similar. By the previous part, the areas of T_i and T'_i are related by $\text{area}(T_i) = k^2\text{area}(T'_i)$. Adding up the areas of all the triangles, we have

$$\text{area}(D) = \sum_1^{n-2} \text{area}(T_i) = k^2 \sum_1^{n-2} \text{area}(T'_i) = k^2\text{area}(D').$$