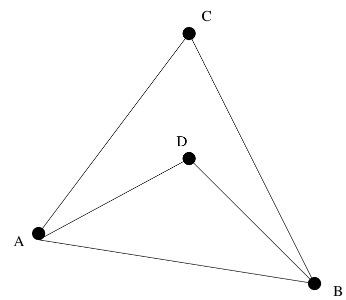
Solutions to the Practice Problems
Math 223
Feb. 14, 2005

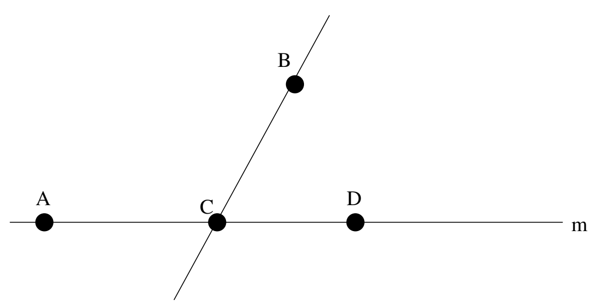
1. For each of Euclid's propositions listed below, draw a picture to illiustrate them and translate them to modern language.

   (a) If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.
   Here's the picture:



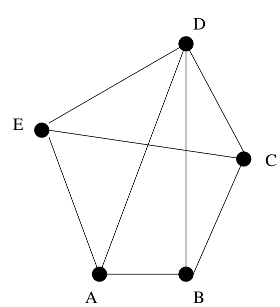
   The statement of the proposition is that in the situation drawn, $|AD| + |BD| < |AC| + |BC|$, while $(\angle ADB)^\circ > (\angle ACB)^\circ$.

   (b) If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.
   Here's the picture: Here's the picture:



   The statement is that if $(\angle ACB)^\circ + (\angle DCB)^\circ = 180^\circ$ then $\vec{CA}$ and $\vec{CD}$ point in opposite directions. The segment CB is the first straight line Euclid mentions, while CA and CD are the constructed segments.

2. Consider the following model with points and lines given. So the points are A, B, C, D, E and the lines are



$\{A, B\}, \{B, C\}, \{C, D\}, \{D, E\}, \{E, A\}, \{A, D\}, \{B, D\}, \{E, C\}$.

(a) Does this satisfy the incidence axioms? List the axioms it satisfies and the one it doesn't. Be sure to explain your answer.

Every line in this model contains precisely two points, so it satisfies the second and third axioms. (Because there are only two points on a line, any triple of distinct points is not collinear.) However, there is no line connecting A and C, so it doesn't satisfy the first axiom.

(b) Regardless of whether it does satisfy the incidence axioms, we can still find parallel lines. What are all the lines parallel to $\{E, C\}$ and incident to D?

The lines parallel to $\{E, C\}$ through D are $\{A, D\}$ and $\{B, D\}$.

3. (a) Prove that all three-point incidence geometries are isomorphic. (Hint: write out the mapping of points; this forces a certain mapping of lines.)

We have a model three-point geometry, where the points are A, B, C and the lines are $\{A, B\}, \{B, C\}, \{A, C\}$; we will show that any three-point geometry is isomorphic to this one. Pick a three-point geometry, and call its points x, y, z. First, this geometry must have precisely three lines. It has to have at least two lines, because x, y, z are not collinear. Say it has precisely two lines: l_1 connecting x and y and l_2 connecting y and z. Notice that l_1 can't contain z, because the three points are not collinear. Similarly, l_2 can't contain x. Then there must be a line l_3 connecting x and z, which contains only x and z (by the same reasoning). Moreover, there can't be more than three lines, because then you'd have mulitple lines connecting a pair of points, which violates the first incidence axiom. Alright, we're almost done now. We choose the mapping of points which sends A to x, B to y, and C to z. In order that we maintain the incidence relationships, the forces the mapping $\{A, B\}$ to l_1, $\{B, C\}$ to l_2 and $\{A, C\}$ to l_3. One can then check that the mapping preserves incidence. For instance, the lines $\{A, B\}$ and $\{B, C\}$ intersect in the point B, while the lines l_1 and l_2 intersect in the point y.

(b) Given a three-point incidence geometry, how many isomorphisms are there from it to itself? (Hint: how many ways can you re-order three things?)

The easiest way to do this is to count how many ways you can reorder an ordered set of three things, and there are $3! = 6$ ways to do this. (Don't forget to count the trivial reordering, where you leave everything in the original order.) Let's count them: you can reorder (A, B, C) as

$$(A, B, C) \quad (A, C, B) \quad (B, A, C) \quad (B, C, A) \quad (C, A, B) \quad (C, B, A).$$

Each reordering gives you an isomorphism.

Another way to see this is to count the number of isometries of the Euclidean plane which preserve an equilateral triangle. There are 6: three reflections and three rotations.

4. Prove ASA congruence of triangles given SAS congruence of triangles.

We start with two triangles $\triangle ABC$ and $\triangle DEF$, where $\angle CAB \simeq \angle FDE$, $AB \simeq DE$, and $\angle ABC \simeq \angle DEF$. We will show $BC \simeq EF$ and then apply SAS. (The same sort of argument shows $AC \simeq DF$.) In relating BC and EF, there are three possibilities: $|BC| = |EF|$, $|BC| < |EF|$, or $|BC| > |EF|$. If the first case holds, we're done. Now suppose $|BC| < |EF|$, then we can find a point G between E and F so that $|BC| = |EG|$, and then $\triangle ABC \simeq \triangle DEG$ by SAS. In particular, this implies $\angle FDE \simeq \angle CAB \simeq \angle GDE$. However, the ray $\vec{DG}$ lies inside the angle $\angle FDE$, so $(\angle GDE)^\circ < (\angle FDE)^\circ$, which contradicts $\angle FDE \simeq \angle GDE$. So we cannot have $|BC| < |EF|$. A similar agument arrives at a contradiction if $|BC| > |EF|$, and so the only possibility is that $BC \simeq EF$.

5. Prove that Playfair's axiom is equivalent to the converse to the alternate interior angles theorem.

First note the statement of the converse of the alternate interior angle theorem. It is

(*) Suppose two lines l and m are cut by a transversal t. If $l \| m$, then alternate interior angles are congruent.

Suppose Playfair is true. Suppose P is the common point of t and m and Q is the common point of t and l. If A lies on l and $A \neq Q$ then we may construct $\angle QPB$ which is, with $\angle AQP$ alternate interior angles and $m\angle QPB = m\angle PQA$. Then the alternate interior angle theorem says that $l \| m$. But there is only one parallel to l through P. So any alternate interior angle to $\angle PQA$ at P made by a parallel to l must be congruent to $\angle PQA$.

If the converse to the alternate interior angle theorem is true, then there is only one line through P which is parallel to l. To see this, note that there is only one alternate interior angle at P which is congruent to $\angle AQP$.

6. Justify the steps in showing that one can construct an angle measuring $135°$, using a straight-edge and a compass.

Observe that $135 = 90 + 45$, so it suffices to construct a $90°$ angle and a $45°$ angle. We construct a right angle as follows: first use a straight edge to mark a straight line segment, and mark a point A on it. Then use a compass to mark two points B and C on the segment such that $|AB| = |AC|$. Next use the compass to mark two circular arcs centered at B and C, having the same radius $r > |AB|$. These two circular arcs will intersect at a point D, which we connect to A using the straight edge. The angles $\angle DAB$ and $\angle DAC$ are right angles (because $\triangle DAB \simeq \triangle DAC$ by SSS). Also, we can use the compass to mark a point $D' \in AD$ so that $|AD'| = |AB|$. Then $\triangle D'AB$ is an isosceles right triangle, so $(\angle AD'B)° = 45° = (\angle ABD')°$.

7. Consider the triangle $\triangle ABC$, and let AD bisect the angle $\angle CAB$, with $D \in BC$. Prove that AD meets BC at a right angle if and only is $AB \simeq AC$. (Hint: congruent triangles. You may use the fact that $AB \simeq AC$ if and only if $\angle ABC \simeq \angle ACB$, and that the angle sum of a triangle is $180°$. You really don't need to use the angle sum part, but it might make the proof easier.)

   There are really two things we have to prove here: if $AB \simeq AC$ then $AD \perp BC$, and if $AD \perp BC$ then $AB \simeq AC$.

   First assume $AB \simeq AC$. We also have $\angle CAD \simeq \angle BAD$ and $AD \simeq AD$, so $\triangle CAD \simeq \triangle BAD$ by SAS. Thus $\angle CDA \simeq \angle BDA$. However, these two angles are supplementary, so they can only be congruent if they are both right angles.

   Next we assume $AD \perp BC$, so in particular $\angle CDA \simeq \angle BDA$. Also, $\angle CAD \simeq \angle BAD$ and $AD \simeq AD$. So $\triangle CAD \simeq \triangle BAD$ by ASA, which implies $AC \simeq AB$.