Practice Problems

Math 223

Dec. 6, 2004

These problems are not in any particular order. As in the midterm exams, I will list axioms and theorems you should know.

1. Consider the Möbius transformation
$$T(z) = \frac{z - 1/2}{z/2 - 1}.$$
   (a) What is the image of the unit disc D under T?
   (b) What is the image of the real axis (i.e. the x-axis) under T?
   (c) What is the image of the imaginary axis (i.e. the y-axis) under T?
   (d) You might recognize T as an isometry of the Poincaré disc, so it is inversion through a particular circle (which meets the unit circle orthogonally). Which circle is it?

2. Show that a Möbius transformation
$$T(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$
has most two fixed points (i.e. solutions to the equation $T(z) = z$) unless T is the identity transformation.

3. Suppose a Möbuis transformation $T(z)$ maps the unit circle $|z| = 1$ to itself, and that $T(1/3) = 0$. What is $T(3)$? (Hint: Möbius transformations preserve the cross-ratio.)

4. Let $\square ABCD$ be a Saccheri quadrilateral in the Poincaré disc, with $BC \simeq AD$ and $(\angle DAB)^\circ = (\angle ABC)^\circ = 90^\circ$.
   (a) Show that $\angle ADC \simeq \angle DCB$.
   (b) Show that $AB > CD$
   (c) Let M be the midpoint of AB and M' the midpoint of CD. Show that $\angle AMM'$ and $\angle DM'M$ are right angles.

5. Consider the following the following set of points and lines.

   (a) Show this is an incidence geometry.
   (b) Is it Euclidean, hyperbolic, ellipitical, or none of the above?

6. Let $\triangle ABC$ have a right angle at vertx B. Show that AC is the largest side (i.e. $AC > BC$ and $AC > AB$).

7. Using Hilbert's axioms, show that supplements of congruent angles are congruent. In other words, if $\angle ABC \simeq \angle DEF$, $A * B * C'$, and $D * E * F'$ then $\angle CBC' \simeq FEF'$. (Hint: First draw a picture. You might want to use SAS for triangles somewhere.)

8. Recall that a rectangle is a quadrilateral with four right angles. Also, the defect of a triangle $\triangle ABC$ is
$$\delta ABC := 180^\circ - (\angle A)^\circ - (\angle B)^\circ - (\angle C)^\circ.$$
Prove that if a rectangle exists then there exists a right triangle with a defect of 0°.

9. Show that the defects of triangles add. In other words, if $A * D * B$ and C is not collinear with A, D, B then

$$\delta ADC + \delta BDC = \delta ABC.$$

10. Suppose there exists exists a line l and a point $P \notin l$ with two distinct lines m_1, m_2 incident to P which are parallel to l. Prove that there are infinitely many lines incident to P which are parallel to l.

11. A parallelogram $\square ABCD$ is a quadrilateral such that pairs of non-adjacent sides (e.g. AB and CD form one pair, while BC and AD form the other pair) lie on parallel lines. In the Euclidean plane, prove that the diagonal AC divides a parallelogram $\square ABCD$ into congruent triangles. (Hint: alternate interior angles) Does this remain true in the hyperbolic plane?

12. Recall the exterior angle theorem: the exterior angle of a triangle is greater that either remote interior angle. Use this to prove SAA for triangles.

13. A regular n-gon is a polygon (either in the Euclidean plane or the hyperbolic plane) such that all its sides and all its interior angles are congruent. Let C be the center of a regular n-gon with vertices $P_1, \ldots, P_n$, and consider the triangles $\triangle P_j C P_{j+1}$. You may assume the radial segments $CP_1, \ldots, CP_n$ all have the same length

    (a) Show that in the Euclidean plane all the triangles $\triangle P_j C P_{j+1}$ are congruent.

    (b) In the Euclidean plane, what is the measure of the angle $\angle P_j C P_{j+1}$? How about the measure of $\angle C P_j P_{j+1}$?

    (c) Answer the same questions in the case of the hyperbolic plane.

14. Show the following two incidence geometries are isomorphic:

    (a) the usual Euclidean plane, with its usual lines

    (b) the sphere except the north pole, with the lines being the intersection of the sphere and any plane in $\mathbb{R}^3$ passing through the north pole

    (Hint: stereographic projection)