# Solutions to the Practice Problems

Math 223

Dec. 9, 2004

1. Consider the Möbius transformation
$$T(z) = \frac{z - 1/2}{z/2 - 1}.$$
   (a) What is the image of the unit disc $D$ under $T$?

   There are a number of ways to find the image of the unit circle. First observe that the image is either a circle or a line, because Möbius transformations always send circles and lines to circles and lines. Also, $T(2) = \infty$, so $T$ sends all points on the unit circle to finite points. Thus the image of the unit circle is a circle. Also, $T(1/2) = 0$. Now use the fact that $1/2$ and $2$ are inverses of each other under inversion through the unit circle. $T$ sends these two points to two other points ($T(1/2) = 0$, and $T(2) = \infty$) which are also inverses of each other in the unit circle. Putting all this together, we see that $T$ maps the unit circle to itself.

   Alternatively, one can show directly that if $|z| = 1$ then $|T(z)| = 1$. First let's see this suffices to show that $T$ maps the unit disc to itself. Because $T$ maps the unit circle to itself, $T$ must map the unit disc to either itself, or to the exterior of the unit disc. However, $T(0) = 1/2$, which is a point inside the unit disc, which implies $T$ maps the unit disc to itself. Ok, now let's show that if $|z| = 1$ then $|T(z)| = 1$. First note that $1 = |z|^2 = z\bar{z}$. So
$$\begin{aligned} |T(z)|^2 &= \frac{|z-1/2|^2}{|z/2-1|^2} = \frac{(z-1/2)(\bar{z}-1/2)}{(z/2-1)(\bar{z}/2-1)} \\ &= \frac{|z|^1 - z/2 - \bar{z}/2 + 1/4}{|z|^1/4 - z/2 - \bar{z}/2 + 1} = \frac{5/4 - z/2 - \bar{z}/2}{5/4 - z/2 - \bar{z}/2} = 1. \end{aligned}$$

   (b) What is the image of the real axis (i.e. the $x$-axis) under $T$?

   For any real number $z \neq 2$, the ratio
$$T(z) = \frac{z - 1/2}{z/2 - 1}$$
   is again a real number. (Observe that $z = 2$ gets sent to the point at infinity.) So the image of the real axis is again the real axis.

   (c) What is the image of the imaginary axis (i.e. the $y$-axis) under $T$?

   We use the same reasoning as above to conclude that the image of the imaginary axis is a circle in the plane. Moreover, we have $T(0) = 1/2$ and $T(\infty) = 2$. Thus the image circle is a circle which meets the unit circle at right angles, and contains $1/2$ and $2$ in such a way that these two points are interchanged under inversion through the unit circle. This is the circle with center $5/4$ (the average of $1/2$ and $2$) and radius $3/4$.

   (d) You might recognize $T$ as an isometry of the Poincaré disc, so it is inversion through a particular circle (which meets the unit circle orthogonally). Which circle is it?

2. Show that a Möbius transformation
$$T(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$
   has most two fixed points (i.e. solutions to the equation $T(z) = z$) unless $T$ is the identity transformation.

   Let $T$ be a Möbius transformation which is not $T(z) = z$. Then fixed points satisfy
$$z = T(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \Leftrightarrow \alpha z + \beta = \gamma z^2 + \delta z.$$
   This latter formula is a quadratic in $z$:
$$0 = \gamma z^2 + (\delta - \alpha) z - \beta,$$
   so it has precisely two solutions (counting multiplicities).

3. Suppose a Möbuis transformation $T(z)$ maps the unit circle $|z| = 1$ to itself, and that $T(1/3) = 0$. What is $T(3)$? (Hint: Möbius transformations preserve the cross-ratio.)

Recall that $z$ and $z^*$ are inverses of each other in a circle (or a line, for that matter) if and only if $(z, z_1, z_2, z_3) = (z^*, z_1, z_2, z_3)$ for any distinct triple of points $z_1, z_2, z_3$ on the circle. The fact that $T$ preserves the cross-ratio tells you that $T$ maps inverse points to inverse points. Note that $1/3$ and $3$ are inverses of each other with respect to the unit circle, and that $T(1/3) = 0$. Thus $T$ must send 3 to the inverse of 0 with respect to the unit circle, which is the point at infinity.

4. Let $\square ABCD$ be a Saccheri quadrilateral in the Poincaré disc, with $BC \simeq AD$ and $(\angle DAB)^\circ = (\angle ABC)^\circ = 90^\circ$.

   This is one of the places where I wish I could draw some pictures. So it may help you to draw the pictures described below.

   (a) Show that $\angle ADC \simeq \angle DCB$.

   First consider $\triangle DAB$ and $\triangle CBA$. We have $\angle DAB \simeq \angle CBA$, $AD \simeq BC$, and $AB \simeq AB$. This implies $\triangle DAB \simeq \triangle CBA$ by SAS. Now, the congruence of these two triangles implies $AC \simeq BD$ and $\angle DAC \simeq \angle CBD$. Combine this with $CD \simeq CD$ to show $\triangle ACD \simeq \triangle BDC$ by SAS. Or, one can use SSS to show $\triangle ACD \simeq \triangle BDC$. Regardless, this shows $\angle ADC \simeq \angle BCD$.

   (b) Show that $AB > CD$

   (c) Let $M$ be the midpoint of $AB$ and $M'$ the midpoint of $CD$. Show that $\angle AMM'$ and $\angle DM'M$ are right angles.

   First, $\triangle DAM \simeq \triangle CBM$ by SAS ($AD \simeq BC$, $AM \simeq BM$ and $\angle DAM \simeq \angle CBM$). Thus $DM \simeq CM$, which implies $\triangle DMM' \simeq CMM'$ by SSS. This gives us $\angle AMD \simeq \angle BMC$ and $\angle DMM' \simeq \angle CMM'$, which implies $\angle AMM' \simeq \angle BMM'$. However, $\angle AMM'$ and $\angle BMM'$ are supplementary, so this is only possible if each is a right angle. The proof that $\angle DM'M$ (and $\angle CM'M$) are right angles is exactly the same.

5. Consider the following the following set of points and lines.

   (a) Show this is an incidence geometry.

   We have to verify the three incidence axioms. There are four points: $A, B, C, D$; and six lines: $\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}$. First, every pair of points uniquely determines a line. In fact, every pair of points is precisely a line. Second, every line contains at least two points. In fact, every line contains precisely two points. Third, there exists three distinct points which are not collinear. Any choice of three distinct points suffices.

   (b) Is it Euclidean, hyperbolic, ellipitical, or none of the above?

   This model is Euclidean. Consider a line $l$; it contains precisely two points $P_1, P_2$. Then for any point $P \notin l$ (there are two such points), there is one line incident to $P$ which does not intersect $l$. This line is $l' = \{P, Q\}$ where $P, Q, P_1, P_2$ are distinct.

6. Let $\triangle ABC$ have a right angle at vertex $B$. Show that $AC$ is the largest side (i.e. $AC > BC$ and $AC > AB$).

   Note that
   $$(\angle A)^\circ \leq 180^\circ - (\angle B)^\circ - (\angle C)^\circ < 180^\circ - (\angle B)^\circ = 90^\circ.$$
   By the same reasoning, $(\angle C)^\circ < 90^\circ$. So $\angle B$ is the largest angle of $\triangle ABC$, which implies that the side opposite $\angle B$, which is $AC$, is the largest side.

7. Using Hilbert's axioms, show that supplements of congruent angles are congruent. In other words, if $\angle ABC \simeq \angle DEF$, $A * B * C'$, and $D * E * F'$ then $\angle CBC' \simeq FEF'$. (Hint: First draw a picture. You might want to use SAS for triangles somewhere.)

   Draw the two angles. Without loss of generality, we can assume $AB \simeq DE$ and $BC \simeq EF$. Then $\triangle ABC \simeq \triangle DEF$ by SAS, so $AC \simeq DF$ and $\angle BAC \simeq EDF$. Also, without loss of generality we can choose $BC' \simeq EF'$. Then $AC' \simeq DF'$ by segment addition, so $\triangle ACC' \simeq DFF'$ by SAS. Then $\angle BC'C \simeq \angle EF'F$, $CC' \simeq FF'$, and $\angle C'CA \simeq \angle F'FD$. This last relation implies $\angle C'CB \simeq \angle F'FE$, and so $\triangle C'CB \simeq F'FE$ by ASA. Finally, this implies $\angle CBC' \simeq FEF'$, which is the conclusion we want.

8. Recall that a rectangle is a quadrilateral with four right angles. Also, the defect of a triangle $\triangle ABC$ is
   $$\delta ABC := 180^\circ - (\angle A)^\circ - (\angle B)^\circ - (\angle C)^\circ.$$
   Prove that if a rectangle exists then there exists a right triangle with a defect of $0^\circ$.

9. Show that the defects of triangles add. In other words, if $A * D * B$ and $C$ is not collinear with $A, D, B$ then
$$\delta ADC + \delta BDC = \delta ABC.$$
Observe that $\angle ADC$ and $\angle BDC$ are supplementary, so
$$(\angle ADC)^\circ + (\angle BDC)^\circ = 180^\circ.$$
Also observe that the ray $\vec{CD}$ is between $\vec{CA}$ and $\vec{CB}$, so
$$(\angle ACD)^\circ + (\angle BCD)^\circ = (\angle ACB)^\circ.$$
Now compute the sum of the defects:
$$\begin{aligned}\delta ADC + \delta BDC &= 180^\circ - (\angle ADC)^\circ - (\angle DAC)^\circ - (\angle ACD)^\circ + 180^\circ - (\angle BDC)^\circ - (\angle BCD) \circ -(\angle CBD)^\circ \\ &= 360^\circ - (\angle ADC)^\circ - (\angle BDC)^\circ - (\angle ACD)^\circ - (\angle BCD)^\circ - (\angle DAC)^\circ - (\angle DBC)^\circ \\ &= 180^\circ - (\angle ACB)^\circ - (\angle BAC)^\circ - (\angle ABC)^\circ = \delta ABC.\end{aligned}$$

10. Suppose there exists exists a line $l$ and a point $P \notin l$ with two distinct lines $m_1$, $m_2$ incident to $P$ which are parallel to $l$. Prove that there are infinitely many lines incident to $P$ which are parallel to $l$.

    We will use the fact that the parallel postulate is equivalent to the existence of rectangles (quadrilaterals with four right angles). So in our case there do not exist any rectangles. Start with $P \notin l$ and $m \| l$, with $m$ incident to $P$. First drop a perpendicular from $P$ to $l$, which intersects $l$ at $Q$, and let $R$ be any point on $l$ which is distinct from $Q$. Now let $t$ be perpendicular to $l$, with $R \in t$, and drop a perpendicular from $P$ to $t$, which meets $t$ at $S$. We make two observations: first, the line $\hat{m} = \vec{PS}$ is parallel to $l$ (because both $l$ and $\hat{m}$ are perpendicular to $t$), and second, the quadrilateral $\Box PQRS$ has at least three right angles (the angles at $Q$, $R$, and $S$). Now suppose $m = \hat{m}$. If this is the case then $m = \hat{m}$ meets $\vec{PQ}$ at right angles and $\Box PQRS$ is a rectangle, which is a contradiction. So $m \neq \hat{m}$. To obtain infinitely many lines, vary $R$. With each different choice of $R$ you obtain a different parallel line $\hat{m}$.

11. A parallelogram $\Box ABCD$ is a quadrilateral such that pairs of non-adjacent sides (e.g. $AB$ and $CD$ form one pair, while $BC$ and $AD$ form the other pair) lie on parallel lines. In the Euclidean plane, prove that the diagonal $AC$ divides a parallelogram $\Box ABCD$ into congruent triangles. (Hint: alternate interior angles) Does this remain true in the hyperbolic plane?

    We know that $\vec{AB} \| \vec{CD}$ and $\vec{AD} \| \vec{BC}$. Also, the diagonal is a transversal between each pair of parallel sides. So the the converse to the alternate interior angle theorem implies $\angle ACB \simeq \angle CAD$ and $\angle CDA \simeq \angle BAC$. Also, $AC \simeq AC$. Thus $\triangle ABC \simeq \triangle CDA$ by ASA. This result is false in the hyperbolic plane, because the converse to the alternate interior angle theorem is equivalent to the parallel postulate.

12. Recall the exterior angle theorem: the exterior angle of a triangle is greater that either remote interior angle. Use this to prove SAA for triangles.

    Consider two triangles $\triangle ABC$ and $\triangle DEF$ with $\angle CAB \simeq \angle FDE$, $\angle ABC \simeq \angle DEF$ and $BC \simeq EF$. If $AB \simeq DE$, then we can use use SAS (or ASA) to conclude $\triangle ABC \simeq \triangle DEF$. So assume either $AB < DE$ or $AB > DE$. In the case $AB < DE$, there is a point $G$ so that $D * G * E$ and $GE \simeq AB$. Then $\triangle ABC \simeq \triangle GEF$ by SAS, so $\angle FGE \simeq \angle CAB \simeq \angle FDE$. However, $\angle FDE$ is a remote interior angle for $\angle FGE$, so by the exterior angle theorem $\angle FDE < \angle FGE$, which gives us a contradiction. In the case that $AB > DE$, there exists a point $G$ so that $A * G * B$ and $GB \simeq DE$. Now argue as before.

13. A regular $n$-gon is a polygon (either in the Euclidean plane or the hyperbolic plane) such that all its sides and all its interior angles are congruent. Let $C$ be the center of a regular $n$-gon with vertices $P_1, \dots, P_n$, and consider the triangles $\triangle P_j C P_{j+1}$. You may assume the radial segments $CP_1, \dots, CP_n$ all have the same length

    (a) Show that in the Euclidean plane all the triangles $\triangle P_j C P_{j+1}$ are congruent.

    The radial segments $P_j C$ are all congruent because $C$ is the center of the polygon. Also, $P_j P_{j+1} \simeq P_i P_{i+1}$ for any choice of $i$ and $j$ because the polygon is regular. Thus $P_j C P_{j+1} \simeq P_i C P_{i+1}$ by SSS. Note that this proof also works in the hyperbolic plane.

    (b) In the Euclidean plane, what is the measure of the angle $\angle P_j C P_{j+1}$? How about the measure of $\angle C P_j P_{j+1}$?

    The angles $P_j C P_{j+1}$ are all congruent, and they sum to $360^\circ$. Thus $(P_j C P_{j+1})^\circ = (360/n)^\circ$. Because $P_j C P_{j+1}$ is an iscosceles triangle, we have $180^\circ = 2(\angle C P_j P_{j+1})^\circ + (360/n)^\circ$. Solving for $(\angle(C P_j P_{j+1})^\circ$ we get $(\angle C P_j P_{j+1})^\circ = 90^\circ - (180/n)^\circ$.

(c) Answer the same questions in the case of the hyperbolic plane.

Our computation of the angle $\angle P_j C P_{j+1}$ is still valid in the hyperbolic plane, so that angle is still $(360/n)^\circ$. However, the angle sum of a triangle is less than $180^\circ$ in the hyperbolic plane, so in this case we get $(\angle C P_j P_{j+1})^\circ < 90^\circ - (360/n)^\circ$.

14. Show the following two incidence geometries are isomorphic:

    (a) the usual Euclidean plane, with its usual lines

    (b) the sphere except the north pole, with the lines being the intersection of the sphere and any plane in $\mathbb{R}^3$ passing through the north pole

    (Hint: stereographic projection)

    Well, it turns out that stereographic projection provides the isomorphism between these two geometries. Let's check that it works. First normalize things by taking the plane to be the horizontal $xy$-plane in $\mathbb{R}^3$ and the sphere to be the standard unit sphere centered at the origin. Recall the definition of stereographic projection: for $p = (x, y, z)$ on the sphere, let $l$ be the line through $p$ and the north pole $N$. Then the image of $p$ under stereographic projection is $\Phi(p) = q = (u, v, 0)$, which is the intersection of $l$ with the $xy$-plane. One can write down formualas for $\Phi$ and $\Phi^{-1}$, but we don't need them. Ok, now look at the image of a "line" on the punctured sphere, where we take lines to be the intersection of the sphere and a plane through $N$, under $\Phi$. Indeed, this image is the intersection of the plane through $N$ and the $xy$-plane, which is a line in the $xy$-plane. Moreover, one can obtain any line in the $xy$-plane this way. Thus $\Phi$ is an isomorphism.