Solutions to the Practice Problems
Math 223
Oct. 31, 2004

1. Recall Euclid's fifth postulate: if a transversal t intersecting two lines l and m so that the angle sum of the interior angles on one side of t is less than $180°$, then the lines l and m intersect, and the intersection point lies on the same side of t as both these angles.

   (a) State the converse to Euclid's fifth postulate.

   Converse: if two lines l and m meet on one side of a transversal t, then the angle sum of the interior angles on that side of t is less than $180°$.

   (b) Prove the converse to Euclid's fifth postulate.

   Let A be the intersection point of l and m, B the intersection point of t and l, and C the intersection point of t and m. Then the interior angles in question are $\angle ABC$ and $\angle ACB$. The triangle $\triangle ABC$ has angle sum at most $180°$, by Saccheri-Legendre. Therefore,

$$180° \geq (\angle ABC)° + (\angle ACB)° + (\angle CAB)° > (\angle ABC)° + (\angle ACB)°,$$

   which implies $(\angle ABC)° + (\angle ACB)° < 180°$.

   (c) **Assuming Euclid's fifth postulate**, prove that if the angle sum of the two interior angles on one side of t is greater than $180°$, then the lines l and m cannot intersect on that side of t.

   Assume by contradiction that l and m meet at a point A on the same side of t as the interior angles in question. We again let B be the intersection point of l and t, and C the intersection point of m and t. Label the supplementary angles to $\angle ABC$ and $\angle ACB$ as α and β, respectively. Then

$$(\alpha)° + (\beta)° = 180° - (\angle ABC)° + 180° - (\angle ACB)° < 180^circ,$$

   which implies (by Euclid V) that l and m have an intersection point A' on the same side of t as α and β. Becuase A and A' are on opposite sides of t, we cannot have $A = A'$. Thus l and m are distinct lines which meet at two points, which is a contradiction.

2. Recall that a Saccheri quadrilateral $\square ABCD$ satisfies $AD \simeq BC$ and $(\angle DAB)° = 90° = (\angle CBA)°$.

   (a) Prove that $\angle ADC \simeq \angle BCD$.

   First consider the triangles $\triangle ABC$ and $\triangle BAD$. We have $AD \simeq BC$, $AB \simeq BA$, and $\angle DAB \simeq CBA$. Thus $\triangle ABC \simeq \triangle BAD$ by SAS. This implie $\angle CAD \simeq \angle DBC$ and $AC \simeq BD$. Then $\triangle ACD \simeq \triangle BDC$, again by SAS. This implies $\angle ADC \simeq \angle\angle BCD$.

   (b) Let M be the midpoint of AB and M' the midpoint of CD. Prove that MM' is perpendicular to both AB and CD.

   First consider $\triangle ADM'$ and $\triangle BCM'$. We have $AD \simeq BC$, $DM' \simeq CM'$, and $\angle ADM' \simeq \angle BCM'$ (by the previous part). Therefore $\triangle ADM' \simeq \triangle BCM'$, and so $AM' \simeq BM'$. Thus $\triangle AMM' \simeq \triangle BMM'$ by SSS, and so $\angle AMM \simeq \angle BMM'$. However, these last two angles are supplementary, so they must be right angles. The proof that $\angle DM'M$ and $\angle CM'M$ are right angles is similar.

   (c) Prove $|MM'| < |AD|$. (Hint: in a triangle, larger sides are opposite larger angles.)

   Here's an easier way to prove $|MM'| < |AD|$, without resorting to triangles. First observe that $(\angle ADM')° < 90°$, because the angle sum of the quadrilateral is strictly less that $360°$ and $\angle ADM' \simeq \angle BCM'$. So there is a ray r emanating from M', which is interior to $\angle MM'D$, such that the angle subtended by r and $\vec{M'M}$ is congruent to $\angle ADM'$. We claim that r intersects the segment AD at some point D'. First let's see why this finishes the proof. Observe that in this case $\square AMM'D'$ is convex quadrilateral with sides perpendicular to its base and equal summit angles. Therefore it is a Saccheri quadrilateral, and $MM' \simeq AD'$. Thus $|MM'| = |AD'| < |AD|$. We complete the proof by showing r intersects AD. This ray is interior to the angle $\angle DM'M$, so bby the Crossbar theorem applied to the triangle $\triangle DM'A$, it must intersect the side AD.

3. Consider a triangle $\triangle ACD$ where $\angle ACD$ is a right triangle, and let B be a point so that $A * B * C$. Prove that $AD > BD > CD$. You might want to draw a picture.

   There are several ways o attack this problem, and here's one using the Pythagorean theorem. The Pythagorean theorem tells us

$$|AC|^2 + |CD|^2 = |AD|^2, \qquad |BC|^2 + |CD|^2 = |BD|^2.$$

The second of these two equations tells us $|BD| > |CD|$, and so $BD > CD$. It remains to show that $|AD| > |BD|$. We have that $|AC| > |BC|$, and so

$$|AD|^2 = |CD|^2 + |AC|^2 > |CD|^2 + |BC|^2 = |BD|^2.$$

Taking squares roots of the resulting inequality ($|AD|^2 > |BD|^2$) gives us the result we want.

4. Recall that the defect of a triangle $\triangle ABC$ is $\delta ABC = 180° - (\angle A)° - (\angle B)° - (\angle C)°$. In the Euclidean plane, all triangles have the same defect (namely, $0°$). Do all triangles in the hyperbolic plane have the same defect? Explain your answer. (Hint: when you sub-divide a triangle, the defects add.)

   No, in the hyperbolic plane the defects of different triangles can be different. Let $\triangle ABC$ be any triangle in the hyperbolic plane, so it has some positive defect $\delta > 0$. Now choose any D such that $A * D * C$. Then

   $$\delta_1 + \delta_2 = \delta ABD + \delta CDB = \delta ABC = \delta.$$

   However, $\delta_1 = \delta ABD$ and $\delta_2 = \delta CDB$. Thus $\delta = \delta_1 = \delta_2$ forces $2\delta = \delta + \delta = \delta$, which is impossible for positive numbers.

5. Let l, l' be distinct parallel lines, with $M \in l$ and $M' \in l$. Prove that if MM' is perpendicular to both l and l' then it is the shortest segment joining l and l'.

   First suppose there is a shorter segment joining l and l' of the form MP, with $M \in l$ and $P \in l'$, $P \neq M'$. Then $\triangle MM'P$ is a right triangle with the right angle $\angle MM'P$. This implies $(\angle M'PM)° < 90°$ (by Saccheri-Legendre), and so MM' is the side opposite a smaller angle than MP. Thus $|MM'| < |MP|$. Similarly, there cannot be a segment of the form $M'Q$, with $Q \in l$ and $Q \neq M$, which is shorter than MM'. Finally, let PQ be any segment joining l and l' with $P \in l'$ and $Q \in l$. Consider the line m perpendicular to l through Q (which exists and is unique). The lines m and l' are not parallel, so m must intersect l' at some point, which we call M'. Also, rename $Q = M$. Then we're in the first situation we examined, and so either $M' = P$ and MM' is perpendicular to both l and l', or $M' \neq P$ and $|MP| > |MM'|$.

6. Prove that the diagonals of a convex quadrilateral intersect. (Hint: the crossbar theorem.)

   Call the vertices of the quadrilateral A, B, C, D, with the sides being AB, BC, CD and DA. By convexity, $\triangle ABD$ lies inside $\square ABCD$. In particular, BD is inside $\square ABCD$. Convexity also implies $\vec{AC}$ is a ray interior to the angle $\angle DAB$. The Crossbar theorem tells us this ray intersects the side BD; call the intersection point E. Then we have either $E = C$, or $A * E * C$, or $A * C * E$. In either of the first two cases we're done. However, $A * C * E$ violates convexity, which completes the proof.

7. The convex hull of a set of points is the smallest convex set containing all of them. Prove that the convex hull of three points A, B, C is the triangle with vertices A, B, C. What are the possibilities for the convex hull of four points?

   Let $\bar{K}$ be the convex hull of A, B, C. Observe that convexity is preserved under intersection (i.e. the intersection of two convex sets is convex; this is immediate from the definition). So another way to describe $\bar{K}$ is to say it's the intersection of all convex sets K containing the points A, B, C. First observe that $\triangle ABC$ is convex by the Crossbar theorem, so $\bar{K} \subset \triangle ABC$. Next let K be any convex set containing the points A, B and C; in particular, this means K contains any line segment jointing points in $\triangle ABC$. However, we can write $\triangle ABC$ as the union of line segments AD, where $B * D * C$, or $D = B$, or $D = C$. In any of these cases, each line segment must lie in K, and so $\triangle ABC \subset K$ for any convex set containing A, B, C. Thus $\triangle ABC \subset \bar{K}$. Because we have shown both these inclusions, we must have $\triangle ABC = \bar{K}$.

   The convex hull for four points can either be a quadrilateral, or a triangle. The easiest way to see this is to take the convex hull of the first three points (which is a triangle), and see where the fourth point lies. Is the fourth point inside the triangle? Then the convex hull is the same triangle. Is the fourth point collinear with two of the three other points? Then the convex hull is a larger triangle. In all remaining cases, the convex hull is a quadrilateral. In general, the convex hull of a finite number of points is a convex polygon, and the number of vertices in the polygon is at most the number of points.

8. Recall Hilbert's parallel postulate: given line L and $p \notin L$, there is at most one line M containing p, parallel to L. Suppose Hilbert's parallel postulate is true, and let k, l, m, n be lines. Prove that if $k \| l$, $m \perp k$ and $n \perp l$ then either $m = n$ or $m \| n$.

   If $m = n$ then we're done. So we assume that $m \neq n$. The fact that $m \perp k$ and $k \| l$ implies that m and l are not perpedicular, so they must meet at some point. Moreover, the converse of the alternate interior angle theorem (which is equivalent to Hilbert's parallel postulate) implies that m meets l at a right angle. So both m and n are perpendicular to l, which implies (by the first corollary to the alternate interior angle theorem) that $m \| n$.