Solutions to the Practice Problems

Math 223

Sept. 17, 2004

1. Negate the following statements.

   (a) P and Q
   negation: $\sim P$ or $\sim Q$

   (b) P implies Q
   negation: $\sim P$ and Q

   (c) Every line contains at least three points.
   negation: There exists a line with less than three points.

   (d) Given a triangle with vertices A, B, C such that $AB \simeq AC$, the two angles with vertices B and C are congruent.
   negation: There exist a triangle with vertices A, B, C such that $AB = AC$ and yet the angles with vertices B and C are unequal.

   (e) If two distinct line l and m intersect, they have only one point in common.
   There exist two distinct lines l, m such that $l \cap m$ contains at least two points.

2. Prove or find a counter-example: All four-point incidence geometries are isomorphic.

   This is false. First we have a four-point geomettry with points A, B, C, D and lines given by all doubleton pairs. Then we have a four-point geometry with points given by A, B, C, D and lines $\{A, B\}$, $\{B, C\}$, $\{B, D\}$ and $\{A, D, C\}$. Both satisfy the axioms of incidence geometry, but they have a different number of lines (six lines vs. four lines) Thus they cannot be isomorphic.

3. Does the following model satisfy the axiom of incidence geometry?

   No, because there is no line incident to both A and C.

4. Consider the following model for an incidence geometry.

   (a) Does this model satisfy the Euclidean parallel postulate?
   No. Consider the line BE and the point D. There are two lines (AD and CD) through D which don't intersect BE.

(b) How many automorphisms (i.e. self-maps which are isomorphisms) does this model have?

I count 8 automorphisms. First observe that any automorphism must send E to itself, because E is the only point with 4 lines running into it. All other points have 2 lines incident. The easiest way to see the automorphisms is to arrange A, B, C, D as the vertice of a square in the plane. There are 4 rotations which preserve the square (by integer multiples of $\pi/2$), and four reflections (through the two diagonals, and the two bisector of the sides). That's it.

5. Consider the following model for an incidence geometry.

Never mind this problem. What I wrote down is not a projective plane. Sorry.

(a) Show this is a projective plane.

(b) Draw a model for the dual projective plane. Be sure to write a sentence or two explaining your answer.

6. Prove that any projective plane has at least seven points.

First, any incidence geometry has three distinct points A, B, C These three points must have three lines linking them (because that's the only 3-point incidence geometry). Basically, these three points must embed as a standrad three point geometry inside the projective plane, which forces at least three lines. Because any line in a projective plan has at least three points, the three line must collectively contain at least six points. Call these new points D, E, F. Now draw a line joining A to D. (There's nothing particular about this choice; I could have equally well chosen B and F.) This cannot be one of the original lines we had, because otherwise the uniqueness part of the first incidence axiom would fail. Thus we have at least one more line, which forces at least one more point (because all line contain at least three points), which brings us to seven total points.

7. Prove that two triangles with congruent sides are congruent, using Hilbert's axioms. (Hint: congruence axiom six tells you SAS holds, so you're done if you can show SSS implies SAS.)

Let's take the triangles $\triangle ABC$ and $\triangle DEF$, with $AB \simeq DE$, $BC \simeq EF$, and $AC \simeq DF$. By congruence 4, there exists a unique ray $\vec{EG}$, emanating from E, on the same side of $\vec{DE}$ as F, such that $\angle ABC \simeq \angle DEG$. Moreover, we can choose $EG \simeq BC$ without loss of generality. Then by SAS we have $\triangle ABC \simeq \triangle DEG$. In comparing $\angle DEF$ and $\angle DEG$, there are three possibilities. If $\angle DEF \simeq \angle DEG$, then $\triangle DEF \simeq \triangle DEG$, and so $\triangle DEF \simeq \triangle ABC$.

If $\angle DEF < \angle DEG$ then $\vec{EG}$ intersect DF in a point G' by the Crossbar Theorem (see the next problem). In this case, we have $\angle CAB \simeq \angle G'DE$ (by the previous congruence), and so the triangles $\triangle ABC$ and $\triangle G'DE$ are congruent by ASA. But this contradicts that $\angle DEG' = \angle DEG < \angle DEF$, because $\angle DEF = \angle ABC$.

In the case that $\angle DEG > \angle DEF$, consider the ray $\vec{DF}$. There is a point G' in this ray such that $D * F * G'$, and then we can argue as before.

8. Let $\vec{AD}$ be a ray between the rays $\vec{AB}$ and $\vec{AC}$. Prove that $\vec{AD}$ intersects the segment BC.

This is known as the Crossbar Theorem. Suppose on the contrary that $\vec{AD}$ does not intersect the segment BC. Let $\vec{AF}$ be the ray opposite $\vec{AD}$ (so that $F * A * D$). If $\vec{AF}$ intersects BC at some point P, then $B * P * C$. But this contradicts the fact that $\vec{AD}$ is interior to the angle $\angle BAC$. The only remaining possibility is that the line through A and D does not intersect BC, which implies that B and C are on the same side of this line. Now chose E so that $E * A * C$, which implies that C and E are on opposite sides of the line through A and D. This together with B and C being on the same side implies that B and E are on opposite sides, But this contradicts that B is in the interior of $\angle DAE$ (because B and E must be on the same side), and so we conclude that $\vec{AD}$ must intersect the segment BC.

9. Prove that if $A * B * C$ then $\vec{BA} \cap \vec{BC} = \{B\}$.

First,

$$\vec{BA} = \{P : A * P * B \text{ or } P * A * B\} \cup \{A, B\}, \qquad \vec{BC} = \{P : C * P * B \text{ or } P * C * B\} \cup \{B, C\}.$$

Let $P \in \vec{BA} \cap \vec{BC}$. If $A * P * B$, or $A = P$, or $P * A * B$, then $P * B * C$, and so $P \notin \vec{BC}$. If $B * P * C$, or $P = C$, or $B * C * P$, then $P \notin \vec{BA}$. Thus the only possibility remaining is that $P = B$.

10. Let $\mathbb{Q}^2$ be the ordered pairs of rational points:

$$\mathbb{Q}^2 := \{(p, q) : p, q \text{ ar rational}\}.$$

(This is called the rational plane.) Show that the first congruence axiom and the elementary continuity axioms both fail.

First congruence axiom: Let $A = (0,0) = A'$ and $B = (1,1)$. Also, let r be the ray emanating from $A' = (0,0)$ and going through $(1,0)$. If there did exist $B' \in r$ such that$AB \simeq A'B'$ then it would have the form $B' = (b,0)$. Also, because the two segments would have the same length,

$$b = |A'B'| = |AB| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

There is no such point in $\mathbb{Q}^2$.

Elmentary continuity axiom: Consider the line through the points $(0,0)$ and $(1,1)$ and the circle centered at $(0,0)$ of radius 1. Any intersection point in both the segment and the circle would have the form (p,p), with p a rational number, such that

$$1 = \mathrm{dist}((p,p),(0,0)) = \sqrt{p^2 + p^2} = p\sqrt{2}.$$

Thus $p = 1/\sqrt{2}$, which is not a rational number.