Selected Solutions to the Third Homework

Math 223

Oct. 18, 2004

1. (problem 1, page 134) This is false. Indeed, any triangle in the Euclidean plane has defect zero, and yet there are many such triangles which are not congruent.

2. (problem 2, page 134) This is true. It is indeed a theorem (following from Hilbert's congruence axioms) that all right angles are congruent.

3. (problem 4, page 135) This is false. The Saccheri-Legendre theorem tells us that the angle sum of a triangle is less than or equal to 180 degrees. It leaves open the possibility that the angle sum is always precisely 180 degrees, or that the angle sum is always less than 180 degrees.

4. (problem 13, page 135) This is true, and we proved it using Hilbert's congruence axioms.

5. (problem 3, page 136) The converse to Euclid's fifth postulate is: suppose two lines l and l' intersect on one side of a transversal t. Then the sum of the angles formed by the lines l and l' and t, lying on the same side of t as the intersection point of l and l' is less than 180 degrees.

   Proof: Let A be the intersection point of l and t, b be the intersection point of l' and t, and C be the intersection point of l and l'. By the Saccheri-Legendre theorem, the angle sum $(\angle A)^\circ + (\angle B)^\circ + (\angle C)^\circ \leq 180^\circ$. Because $(\angle C)^\circ > 0^\circ$, we must have $(\angle A)^\circ + (\angle B)^\circ < 180^\circ$. This is the result we wanted to prove.

6. (problem 13, page 138)

   (a) We want to show that midpoints of a segment are unique.

   Let AB be a segment, and suppose C, C' are both midpoints. We have three cases: $A*C*C'$, $A*C'*C$, or $C = C'$. In the first case, we have $|AC| < |AC'|$, and in the second case we have $|AC'| < |AC|$. However, by the definition of midpoints we also have

   $$|AC| = \frac{1}{2}|AB| = |AC'|.$$

   This eliminates the first two possibilities, and so we must have $C = C'$.

   (b) We want to show that any angle has a unique bisector.

   Let the angle be bound by the rays $\vec{AB}$ and $\vec{AC}$. Without loss of generality, we can take $|AB| = |AC|$. Now let M be the midpoint of the segment BC. Then the triangles $\triangle ABM$ and $\triangle ACM$ are congruent by SSS. This implies $\angle BAM \simeq \angle CAM$, and so the ray $\vec{AM}$ bisects the angle $\angle BAC$.

   Next we show uniqueness. Suppose there are two rays $\vec{AM}$ and $\vec{AM'}$ which bisect $\angle BAC$. Again, we can take $|AB| = |AC|$ and $|AM| = |AM'|$, so this time $\triangle ABM \simeq \triangle ABM'$ by SAS. This implies $\angle BAM \simeq \angle BAM'$. By a similar argument, $\angle CAM \simeq \angle CAM'$, and so $\vec{AM} = \vec{AM'}$, proving uniqueness.

7. (problem 17, page 139)

   (a) We want to prove that the segment connecting the midpoint of a chord is perpendicular to the chord.

   Label the endpoints of the chord A and B, the midpoint M, and center of the circle O. Then OA and OB are both radii, so $OA \simeq OB$. By the definition of midpoints, $AM \simeq BM$. Then $\triangle AMO \simeq \triangle BMO$, by SSS. This means $\angle AMO \simeq \angle BMO$. However, the supplement of $\angle AMO$ is $\angle BMO$, and so $\angle AMO$ is a right angle (it's congruent to its own supplement).

   (b) We want to show that the perpendicular bisector of a chord is the line through O and M, with the same notation as above.

   Observe that the perpendicular bisector of any segment is unique. First take the case where AB does not contain O. In this case, the previous part shows that the line through O and M meets AB in a right angle, and bisects AB. Therefore it is the perpendicular bisector. In the case where O lies on AB, the chord s a diameter, with O as its midpoint. Again, the result follows by definition.