Selected Solutions to the Second Homework

Math 223

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1. (9, page 64)

   (a) "lines" are circles: not an incidence geometry because there are multiple lines through each pair of distinct points.

   This also satisfies the hyperbolic parallel property. Take any circle γ and point $p \notin \gamma$. Then there are many other circles through p which do not intersect γ. In fact, once you find one, you can change its radius.

   (b) "points" are lines in 3-space and "lines" are plane: not an incidence geometry because you can find two skew lines which are not contained in any plane.

   This is also hyperbolic: take parallel planes.

   (c) "points" are lines through 0 in 3-space: this is an incidence geometry. It satisfies the elliptic parallel postulate.

   (d) points inside a circle, "lines" are chords: this is an incidence geometry. It satisfies the hyperbolic parallel postulate

   (e) pairs of antipodal points on a sphere: this in an incidence geometry, and it satisfies the elliptic parallel postulate.

2. (10, page 65)

   (a) Start with geometry 1: points are A, B, C and lines are the doubleton sets $\{A.B\}$, $\{B, C\}$ and $\{A, C\}$. incidence is set membership. Now consider any other 3-point geometry, call it geometry 2. It has points x, y, z. Let l_1 be the line joining x to y, l_2 be the line joining y to z and l_3 be the line joining z to x. By the first incidence axiom, the three points x, y, z are not collinear, so the three lines l_1, l_2, l_3 are distinct. (If any two of the three lines l_1, l_2, l_3 coincided, then x, y, z would all lie on the same line.) Now define the mapping between geometries by

   $$A \mapsto x, \quad B \mapsto y, \quad C \mapsto z.$$

   This forces

   $$\{A, B\} \mapsto l_1, \quad \{B, C\} \mapsto l_2, \quad \{A, C\} \mapsto l_3,$$

   which also preserve incidence. Thus the geometry 1 is isomorphic to geometry 2, which was any three-point incidence geometry.

   (b) No: Take A, B, C, D as points, lines are $\{A, B\}$, $\{A, D, C\}$, $\{B, D\}$ and $\{B, C\}$. This only has 4 lines, vs. the standard exam from the text which has 6 lines. So they cannot be isomorphic.

   (c) The isomorphism is given by mapping the line through the origin to the pair of points which is its intersection with the unit sphere centered at the origin. This pair of points is antipodal because the line goes through the origin.

3. (1, page 65) Let M be a projective plane and define a new geometry M' by letting the points of M' be the lines of M, and the lines of M' be the points of M. Let $x, y \in M'$ be distinct points. Then they are lines in M, and so because M is a projective plane they intersect in precisely one point $A \in M$, which is the unique line in M' joining x, y. Next, let A be any line in M'. It is a point in M, so it is incident to at least two distinct lines x, y, which are two distinct points in M' lying on A. Next, pick any line x' in M. It contains three points A, B, C. There is a fourth point D not lying on x', and lines x, y, z joining D to A, B, C repsectively. However, the lines x, y, z intesect in different points, so in M' they form a non-collinear triple of points. This same construction shows that any line in M has at least three other distinct lines incident to it, which implies that all lines in M' contain at least three points. Finally, any two distinct points A, B in M are joined by a line x, which implies that all pairs of lines A, B in M' intersect in the point x. This completes the proof that M' is a projective plane. Also, it shows that the the mapping described above between M and M' is an isomorphism of incidence geometries. If M (and hence M') has finitely many points, then an isomorphic copy of M, which is M', has the same number of lines. This shows that in the finite case the number of points and lines in M is the same.

4. (5, page 66) We have points on a punctured sphere $S^2 \backslash \{N\}$, and lines are any circle through N. This is a little easier to work with if we transfer everything to the Euclidean plane. Let Π be the horizontal u, v plane and S^2 be the unit sphere centered at $(0, 0, 1)$, i.e.

   $$S^2 = \{(x, y, z) : x^2 + y^2 + (z - 1)^2 = 1\}.$$

Here the north pole is $N = (0,0,2)$, and we can define a one-to-one correspondence of points between Π and $S^2 \backslash \{N\}$ in the following way. To the point $(x, y, z) \in S^2 \backslash \{N\}$, we associate the point $(u, v) = (u, v, 0) \in \Pi$, where the three points N, (x, y, z), and $(u, v, 0)$ are collinear. We can parameterize this line as

$$t \mapsto t(x, y, z) + (1 - t)(0, 0, 2) = (tx, ty, 2 - t(z - 2)).$$

To get a point on Π, we must have $0 = 2 - t(z - 2)$, or $t = 2/(2 - z)$. Thus

$$(u, v) = (\frac{2x}{2 - z}, \frac{2y}{2 - z}).$$

Notice this formula is well-defined, because $z < 2$ for all points in $S^2 \backslash \{N\}$.

The formula isn't too important; rather, the geometric interpretation is the key idea. You can picture the mapping as follows: place a crystal ball on a table top, and paint a dot somewhere on the ball, which is not the north pole. Now shine a light from the north pole and look at where this dot casts a shadow on the table top. The position of this shadow is the image of the dot under the mapping. By the way, this transformation is so important it has a special name: stereographic projection.

Ok, now that we have transfered everything to the plane Π, what are the images of circles through N? These circles are cut out by planes passing through N, and each of these planes extends to cut out a (usual) line in the plane Π. We then conclude, after transfering everything over to Π, that the lines in this model are just the usual lines in the Euclidean plane! So this geometry is the usual Euclidean geometry you're used to, described in a different way. Consequently, it is an incidence geometry, and it satisfies the Euclidean parallel postulate.

5. (6, page 67) We have to find two incidence geometries, call them geometry 1 and geometry 2, such that all the lines in geometry 1 have the same number of points, and geometry 2 has two lines with different numbers of points. We can take the standard 3-point geometry as geometry 1; all its lines have 2 points. For geometry 2, we can take the 4-point geometry with 4 lines listed in part (b) of problem 10.

6. Automorphisms of 3-point geometries.

   There are 6 automorphisms. One way to see the 6: list the three points as A, B, C. Then for the mapping there are three choices for where you send A, two remaining choices for where you send B and then only one choice for where you can send C. This also forces where you send the lines (because the you have to preserve incidence).

7. Any incidence geometry has at least three lines.

   First observe any incidence geometry has at least three points by the third incidence axiom. Call them A, B, C. Also, they are not collinear. So there is a line between l_1 A and B, line l_2 between B and C, and a line l_3 between C and A. The fact that A, B, C are not collinear implies that the lines l_1, l_2, l_3 are distinct. This completes the proof.