Some Properties of Great Circles on Spheres

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We will begin with some definitions, which you may already know. First of all, we will denote the unit sphere by S^2; it is

$$S^2 = \{p \in \mathbb{R}^3 : \text{dist}(p, 0) = 1\}.$$

Here 0 is the origin in $\mathbb{R}^3$.

Definition 1 *A great circle is any curve which is the intersection of S^2 with any plane passing through* 0*.*

One should think of great circles in S^2 as the analog of lines in $\mathbb{R}^2$. Indeed, they locally minimize distance (see below), and have as little curvature as possible. However, they do not satisfy the parallel postulate.

Proposition 1 *Given any great circle l and point $p \in S^2$ such that $p \notin l$, there does not exist a great circle m such that $p \in m$ and $m \cap l = \emptyset$.*

It might help to look at Figure 1.

Proof: By rotating S^2, we can assume that l is the equator (*i.e.*, it is horizontal) and p lies in the upper hemisphere. Any great circle m with $p \in m$ is given by the intersection of S^2 with a plane, which we will call Π. This plane Π is not horizontal, so its normal vector has a horizontal component. Thus it contains points in both the upper and lower hemisphere *i.e.* points with both positive and negative heights. Because the height function is continuous on m, the intermediate value theorem implies that m contains at least one point with height zero. However, l consists of all points in S^2 with height zero, so there must be at one point of intersection between l and m. $\square$

Next we sketch a proof of the fact that great circles are distance minimizing. Let $p, q \in S^2$ be distinct points, and let c be a curve joining them, which is not an arc of a great circle. That is, the endpoints c are p and q. Also, let l be the great circle thorugh p which is tangent to c.

(Technically, this requires c to have some regularity; to discuss this would bring us too far afield.) It might help to look at Figure 2.

That c is not a great circle arc implies it must leave l at some point p_1 between p and q. Now reflect curve c through the plane which contains l; the resulting curve c' will have the same length as c, but it will have a kink. One can smooth out that kink to shorten c', which means that c' did not realize the minimum distance from p to q. Because (by construction) c has the same length as c', we have also shown that c cannot minimize distance.

Notice that this reflection technique does not work for great circle arcs; indeed, these arcs are precisely the arcs preserved by reflection through plane through the center of the sphere.

The fact that great circle arcs are the least curved of all possible curves on the sphere is quite involved. In fact, the subject of curvature could easily fill a year-long course (or two or three such courses). See H. Hopf, *Differential Geometry in the Large* for a gentle introduction.