

## Solutions to the Midterm Exam

Math 223

Nov 4, 2004

1. (a) (5 points) Let  $l$  and  $m$  be distinct lines, both of which are perpendicular to  $t$ . Prove that  $l$  and  $m$  are parallel.

Let  $B$  be the intersection point of  $l$  and  $t$ , and  $B'$  the intersection point of  $m$  and  $t$ . Also, let  $A, C \in l$  with  $A * B * C$ ,  $A', C' \in m$  with  $A' * B' * C'$ , and  $A, A'$  on the same side of  $t$ . Then  $\angle ABB'$  and  $\angle C'B'B$  are both right angles, so they're congruent. Then by the alternate interior angles theorem,  $l \parallel m$ .

- (b) (5 points) As a corollary, prove that the the point  $P$  does not lie on the line  $l$ , then there is at least one line  $m$  through  $P$  which is parallel to  $l$ .

First let  $t$  be the line through  $P$  which is perpendicular to  $l$ . Then let  $m$  be the line through  $P$  which is perpendicular to  $t$ . Then  $t$  is perpendicular to both  $l$  and  $m$ , so by the previous problem  $l \parallel m$ .

2. (10 points) Let  $\gamma$  be a circle with center  $O$ ,  $A, B$  points on  $\gamma$  which are not antipodal (i.e.  $A, B, O$  are not collinear), and  $M$  the midpoint of  $AB$ . Prove that  $AB \perp OM$ .

First,  $OA \simeq OB$  because both are radii of  $\gamma$ . Next,  $AM \simeq BM$  by the definition of midpoints. Also,  $OM \simeq OM$  by congruence 2. Therefore,  $\triangle OMA \simeq \triangle OMB$  by SSS, so  $\angle OMA \simeq \angle OMB$ . However, these angles are also supplementary, so the only way  $\angle OMA$  and  $\angle OMB$  can be congruent is if they are both right angles.

3. Consider a triangle  $\triangle ABC$ .

- (a) (10 points) Prove that in the Euclidean plane (i.e. assuming the parallel postulate) the bisector of the angle  $\angle CAB$  is perpendicular to the side  $BC$  if and only if  $AC \simeq AB$ .

Let  $r$  be the ray which bisects the angle  $\angle CAB$ . By the crossbar theorem,  $r$  intersects the side  $BC$ ; call this intersection point  $D$ . First suppose  $AC \simeq AB$ . In this case, we also have  $\angle CAD \simeq \angle BAD$  and  $AD \simeq AD$ , so  $\triangle ADB \simeq \triangle ADC$  by SAS. Thus the supplementary angles  $\angle CDA$  and  $\angle BDA$  are congruent, which is only possible if they are both right angles. Conversely, suppose  $\angle CDA$  and  $\angle BDA$  are right angles. Then they are congruent because all right angles are congruent. We still have  $\angle CAD \simeq \angle BAD$  and  $AD \simeq AD$ , so  $\triangle CAD \simeq \triangle BAD$  by ASA this time. This implies  $AC \simeq AB$ .

- (b) (10 points) Does this remain true in the hyperbolic plane? Be sure to explain your answer.

Yes, the result holds in the hyperbolic plane. Each of our steps only used results in neutral geometry.

4. (10 points) Let  $\triangle ABC$  be a triangle in the hyperbolic plane and let  $L$  be the midpoint of  $BC$ ,  $M$  the midpoint of  $AC$ , and  $N$  the midpoint of  $AB$ . Prove that  $\triangle ABC$  is not similar to  $\triangle ANM$ . (Hint: look at the sum of the interior angles of  $\square MNBC$ .)

Suppose  $\triangle ANM$  and  $\triangle ABC$  are similar. Then  $\angle ABC \simeq \angle ANM$  and  $\angle ACB \simeq \angle AMN$ . This implies

$$(\angle MNB)^\circ = 180^\circ - (\angle ANM)^\circ = 180^\circ - (\angle ABC)^\circ, \quad (\angle NMC)^\circ = 180^\circ - (\angle AMN)^\circ = 180^\circ - (\angle ACB)^\circ.$$

Adding up the interior angles of  $\square MNBC$ , we get

$$(\angle MNB)^\circ + (\angle NMC)^\circ + (\angle NBC)^\circ + (\angle MCB)^\circ = 180^\circ - (\angle ABC)^\circ + 180^\circ - (\angle ACB)^\circ + (\angle ABC)^\circ + (\angle ACB)^\circ = 360^\circ.$$

However, this contradicts the fact that the angle sum of a convex quadrilateral is strictly less than  $360^\circ$  in the hyperbolic plane. Incidentally, the same proof also shows  $\triangle ABC$  is not similar to  $\triangle AMN$ .