Solutions to the First Midterm Exam

Math 223

Sept. 29, 2004

1. Negate the following statements.

   (a) (5 points) Each pair of distinct lines l and m intersect.

   Negation: There exist a pair of lines l and m which do not intersect.

   (b) (5 points) If the side lengths a, b, c of a triangle satisfy $a^2 + b^2 = c^2$ then one of the angles of the triangle is a right angle.

   There exists a triangle whose side lengths a, b, c satisfy $a^2 + b^2 + c^2$ such that none of its angles are right angles.

2. Consider the following model.

   (a) (6 points) Show this model is an incidence geometry.

   First cyclically order the vertices of the pentagon (labeled A through E, in alaphabetical order, in the picture). Two adjacent vertices in the ordering are joined by a side of the pentagon, and two non-adjacent vertices are joined by a diagonal. This shows incidence 1. Any of these segments (either a side or a diagonal) is incident to (precisely) 2 vertices, which shows incidence 2. Finally, A, B and E are three non-collinear points, showing incidence 3.

   (b) (6 points) List the lines parallel to the line through A and B.

   The lines parallel to the line through A and B are the lines through E and C, through D and E, and through D and C. These are the three lines which do not intersect the line through A and B.

   (c) (6 points) Does this model satisfy the hyperbolic parallel postulate? (hyperbolic parallel postulate: given any line l and point $p \notin l$, there exist at least two lines m_1, m_2 which are parallel to l and contain p.) Hint: you can use the symmetry of the figure to only check a couple of lines and points.

   Yes, it is hyperbolic. When I wrote the solutions, I realized that there is a slick way to prove this which makes my hint superfluous; oh well. First observe that any line contains precisely two points, and that there are five points total. Now let l be any line and p be a point not lying on l. Because l contains precisely two points and there are five points total, there are two more points p', p'' such that (i) p, p', p'' are distinct and (ii) p', p'' also do not lie on l. Now let m_1 be the line joining p to p' and m_2 the line joining p to p''. These are distinct lines which do not intersect l, proving that the hyperbolic parallel postulate holds.

3. Recall that $AB < CD$ if there exists a point E such that $C * E * D$ and $AB \simeq CE$. Similalrly, $AB > CD$ if there exists a point E such that $C * D * E$ and $AB \simeq CE$.

   (a) (6 points) Show that exactly one of $AB < CD$, $AB \simeq CD$, $AB > CD$ holds.

   By congruence 1, there is a unique point E on the ray $\vec{CD}$ such that $AB \simeq CE$. If $E = D$ then $AB \simeq CD$. Otherwise, we must have either $C * E * D$ or $C * D * E$, but not both. In the first case, $AB < CD$ and in the second case $AB > CD$.

   (b) (6 points) Show that if $AB < CD$ and $CD < EF$ then $AB < EF$.

   By congruence 1, there is a unique point G on $\vec{EF}$ such that $EG \simeq AB$. Also, by definition, there is a point H such that $E * H * F$ and $CD \simeq EH$. If $G = H$ then $AB \simeq CD$, which contradicts $AB < CD$. Also, if $E * H * G$ then $AB > CD$, which is another contradiction. The only remaining possibility is $E * G * H$, which implies $E * G * F$, and so $AB < EF$.

4. (10 points) Consider the triangles $\triangle ABC$ and $\triangle DEF$. Show that if $\angle CAB \simeq \angle FDE$, $\angle ABC \simeq \angle DEF$, and $AB \simeq DE$, then $\triangle ABC \simeq \triangle DEF$. (Hint: This is ASA; you know SAS. So you might want to show that another pair of sides is congruent.)

   Let $\vec{AG}$ and $\vec{DH}$ be the opposite rays to $\vec{AC}$ and $\vec{DF}$ respectively. Without loss of generality, we can take $AG \simeq DH$. Also, $\angle GAB$ and $\angle HDE$ are supplements to congruent angles ($\angle CAB$ and $\angle FDE$), so they are congruent. This implies, by SAS, that $\triangle GAB \simeq \triangle HDE$. Then

   $$\angle GBA \simeq \angle HED \Leftarrow \angle GBC \simeq \angle HEF$$

   by congruence 4. Also, $GB \simeq HE$, so $\triangle GBC \simeq \triangle HEF$ by SAS. Therefore, $GC \simeq HF$, and so by segment subtraction (equivalently, congruence 3), $AC \simeq DF$. This implies $\triangle ABC \simeq \triangle DEF$ by SAS.