Practice Problems

Math 221

April 18, 2007

1. Consider the differential equation
$$x' + 2tx = t^2.$$
   (a) Find the general solution. (Hint: use an integrating factor.)
   (b) Find the solution to the initial value problem with $x(0) = -1$.

2. Consider the differential equation $x' = (1 + 2t)(x^2 - 1)$.
   (a) Find all the fixed points (i.e. constant solutions).
   (b) Find the general solution. (Hint: separate variables.)
   (c) Do you recover *all* the solutions from the general solution? Explain your answer. (Hint: try to recover the constant solutions you found from the general solution by solving an initial value problem at $t = 0$.)

3. Consider the differential equation $x' = (e^x - 1)(1 - x^2)$.
   (a) Find all the fixed points (constant solutions).
   (b) Sketch the phase portrait of this differential equation.
   (c) Classify which of these fixed points is stable.

4. Consider the differential equation $x'' - 5x' + 4x = 0$.
   (a) Verify that $x_1 = e^{4t} + e^t$, $x_2 = e^{4t} - e^t$ and $x_3 = 2e^{4t} - 2e^t$ are all solutions to the equation.
   (b) Do $\{x_1, x_2\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)
   (c) Do $\{x_1, x_3\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)

5. Consider a general second order linear differential equation of the form $x'' + p(t)x' + q(t)x = 0$.
   (a) Prove that the space of solutions is a two-dimensional vector space.
   (b) Prove that if x_1 and x_2 are solutions and they are linearly independent at $t = 0$ (i.e. one can write the solution to any initial value problem with $x(0) = a$ and $x'(0) = b$ as a linear combination of x_1 and x_2) then x_1 and x_2 are linearly independent for all t.

6. Consider the differential equation
$$x'' - 6x' + 9x = \sin t.$$
   (a) Find the general solution to the associated homogeneous problem.
   (b) Find a solution to the inhomogeneous problem using your favorite method.
   (c) Find the solution to the inhomogeneous initial value problem with $x(0) = 1$ and $x'(0) = -1$.

7. Consider the differential equation
$$x'' - 2x' + x = e^t.$$
   (a) Find the general solution to the associated homogeneous problem.
   (b) Find a solution to the inhomogeneous problem using your favorite method.
   (c) Find the solution to the inhomogeneous initial value problem with $x(0) = 2$ and $x'(0) = 0$.

8. Consider the system of differential equations
$$u' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} u.$$
   (a) Find the general solution using your favorite method.
   (b) Find the solution to the initial value problem with $u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   (c) What is the stability behavior of the fixed point $u = 0$? Explain your answer.

9. Consider the system

$$u' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} u.$$

   (a) Find the general solution using your favorite method.

   (b) Solve the initial value problem with $u(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   (c) Classify the origin as stable or unstable.

10. Consider a buoy suspended in water, shaped like a cylinder or radius r and height h. Let the buoy have uniform density $\rho \leq 1/2$ and remember that water has density 1. Let $x = x(t)$ be the depth to which the buoy is submerged and suppose $x(0) = 0$.

   (a) The forces acting on the buoy are a gravitational force of mg and a buoyancy force of $\pi r^2 gx$. Use this information to write down a differential equation for x.

   (b) Conclude that the buoy undergoes simple harmonic motion with equilibrium $x_0 = \rho h$ and period $2\pi\sqrt{\rho h/g}$.

   (c) Compute the period and amplitude of the motion when $\rho = .5\text{g/cm}^3$, $h = 200\text{cm}$, and $g = 980\text{cm/s}^2$.

11. Consider the soft spring equation

$$x'' + 4x - x^3 = 0.$$

   (a) Let $v = x'$ and rewrite the second order scalar equation as a first order system in x and v.

   (b) Show that the quantity

$$E = \frac{v^2}{2} + 2x^2 - \frac{x^4}{4}$$

   is conserved by solutions. (Hint: take the derivative of E with respect to t). Conclude that solutions must lie on curves in the (x, v) plane where E is constant.

   (c) Show that the only equilibria are $(x, v) = (0, 0)$, $(x, v) = (2, 0)$ and $(x, v) = (-2, 0)$.

   (d) Linearize the system about each equilibrium and classify the point as stable or unstable.

   (e) Sketch some representative solution curves.

12. Consider the one-parameter family of differential equations with parameter α given by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} u_1 - \alpha u_1 u_2 \\ u_2 + \alpha u_1 u_2 \end{pmatrix} = F(u_1, u_2).$$

   (a) Verify that the only fixed points of this system are $(0, 0)$ and $(-1/\alpha, 1/\alpha)$ for $\alpha \neq 0$, and only $(0, 0)$ for $\alpha = 0$.

   (b) Linearize this system about each fixed point.

   (c) What is the stability behavior of each fixed point? Explain your answer.

   (d) Draw some representative phase portraits.

   (e) Is there a bifurcation point in α? If there is, find it. Explain your answer.

13. Consider the one-parameter family of differential equations with parameter α given by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} (e^{u_1} - 1)(\alpha - u_2) \\ (e^{u_2} - 1)(\alpha - u_1) \end{pmatrix} = F(u_1, u_2).$$

   (a) Verify that the only fixed points of this system are $(0, 0)$ and (α, α).

   (b) Linearize this system about each fixed point.

   (c) What is the stability behavior of each fixed point? Explain your answer.

   (d) Draw some representative phase portraits.

   (e) Is there a bifurcation point in α? If there is, find it. Explain your answer.

14. Compute the Laplace transform of the function $f(t) = te^t$.

15. (a) If $g(t) = f(-t)$, show that $\mathcal{L}(g)(s) = -\mathcal{L}(f)(s)$.

   (b) If $g(t) = f(at)$ (where $a > 0$ is a constant), show that $\mathcal{L}(g)(s) = \frac{1}{a}\mathcal{L}(f)(s/a)$.

16. Recall that for a function f to be of exponential order it means that $\lim_{t\to\infty} e^{-ct}|f(t)| = 0$ for some choice of c.

    (a) Show that $f(t) = \cos(e^{t^2})$ is of exponential order. (In fact, it is bounded, which is much better than exponential order.)

    (b) Show that f' is not of exponential order.

17. (a) Solve the initial value problem

$$x'' + tx' + x = 0 \quad x(0) = 1 \quad x'(0) = 0.$$

    (Hint: try taking the Laplace transform of both sides.)

    (b) Solve the initial value problem

$$x'' + e^t x' - x = 0 \quad x(0) = -1 \quad X'(0) = 1.$$

    (Hint: try taking the Laplace transform of both sides.)