Practice Problems

Math 221

April 18, 2007

1. Consider the differential equation
$$x' + 2tx = t^2.$$
   (a) Find the general solution. (Hint: use an integrating factor.)
   In this case the integrating factor is
   $$\mu = e^{\int 2t dt} = e^{t^2}.$$
   Multiplying the equation by μ we get
   $$t^2 e^{t^2} = e^{t^2} x' + 2te^{t^2} x = \frac{d}{dt}(xe^{t^2}) \Rightarrow x(t)e^{t^2} = c + \int_0^t \tau^2 e^{\tau^2} d\tau.$$
   We can't evaluate the indefinite integral in closed form, but that's fine. Anyhow, solving for $x(t)$ we have
   $$x(t) = e^{-t^2}(c + \int_0^t \tau^2 e^{\tau^2} d\tau)$$
   for some constant c.

   (b) Find the solution to the initial value problem with $x(0) = -1$.
   We plug $t = 0$ into the formula above to get
   $$-1 = x(0) = c \Rightarrow x(t) = e^{-t^2}(-1 + \int_0^t \tau^2 e^{\tau^2} d\tau).$$

2. Consider the differential equation $x' = (1 + 2t)(x^2 - 1)$.
   (a) Find all the fixed points (i.e. constant solutions).
   The fixed points satisfy
   $$0 = x' = (1 + 2t)(x^2 - 1).$$
   Because this must hold for all t, the only solutions are $x = \pm 1$, where $x^2 - 1 = 0$.

   (b) Find the general solution. (Hint: separate variables.)
   We can separate variables:
   $$\frac{1}{x^2 - 1} \frac{dx}{dt} = 1 + 2t.$$
   Integrating both sides of the equation with respect to t we have
   $$c + t + 2t = \int \frac{dx}{x^2 - 1} = \frac{1}{2} \int \frac{1}{x - 1} - \frac{1}{x + 1} dx = \frac{1}{2} \log \left(\frac{x - 1}{x + 1} \right).$$
   If we take exponentials and solve for x we get
   $$x = \frac{1 + ce^{2t + 2t^2}}{1 - ce^{2t + 2t^2}}$$
   for some constant c.

   (c) Do you recover *all* the solutions from the general solution? Explain your answer. (Hint: try to recover the constant solutions you found from the general solution by solving an initial value problem at $t = 0$.)
   No, you don't recover the fixed points from the general solution. You can see this by trying to set $x(0) = \pm 1$ and solving for c; there is no solution to the equation you get for c.

3. Consider the differential equation $x' = (e^x - 1)(1 - x^2)$.
   (a) Find all the fixed points (constant solutions).
   As before, the fixed points satisfy
   $$0 = x' = (e^x - 1)(1 - x^2).$$
   If $e^x - 1 = 0$ this forces $x = 0$. If $1 - x^2 = 0$ this forces $x = \pm 1$. These are all the fixed points.

   (b) Sketch the phase portrait of this differential equation.

(c) Classify which of these fixed points is stable.
There are several ways to classify the fixed points, here we look at the sign of the right hand side. When $x < -1$ the right hand side $f(x) = (e^x - 1)(1 - x^2)$ is positive, so the solutions are increasing. Meanwhile, when $-1 < x < 0$ the right hand side f is negative, so solutions are decreasing. Putting this together we see $x = -1$ is stable. On the other hand when $0 < x < 1$ the right hand side f is positive, so solutions are increasing. Together with the fact that solutions are decreasing for $-1 < x < 0$ we see $x = 0$ is unstable. Finally, we see that for $x > 1$ the right hand side f is negative, so solutions are decreasing. Putting this together with the fact that solutions are increasing for $0 < x < 1$ we see that $x = 1$ is stable.

4. Consider the differential equation $x'' - 5x' + 4x = 0$.

(a) Verify that $x_1 = e^{4t} + e^t$, $x_2 = e^{4t} - e^t$ and $x_3 = 2e^{4t} - 2e^t$ are all solutions to the equation.
Plug these functions into the differential equation to see they work.

(b) Do $\{x_1, x_2\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)
Evaluate:

$$x_1(0) = 2, \qquad x_1'(0) = 5, \qquad x_2(0) = 0, \qquad x_2'(0) = 3.$$

Then the Wronskian (evaluated at $t = 0$) is

$$W(x_1, x_2)(0) = x_1(0)x_2'(0) - x_2(0)x_1'(0) = 2 \cdot 3 - 0 \cdot 5 = 6 \neq 0,$$

so x_1 and x_2 do span the solution space.

(c) Do $\{x_1, x_3\}$ form a basis for the solution space? Justify your answer. (Hint: the Wronskian.)
Again we evaluate:

$$x_1(0) = 2, \qquad x_1'(0) = 5, \qquad x_3(0) = 4, \qquad x_3'(0) = 6.$$

Then the Wronskian is

$$W(x_1, x_3)(0) = x_1(0)x_3'(0) - x_3(0)x_1'(0) = 2 \cdot 6 - 4 \cdot 5 = -8 \neq 0,$$

so x_1 and x_3 also span the solution space.

5. Consider a general second order linear differential equation of the form $x'' + p(t)x' + q(t)x = 0$.

(a) Prove that the space of solutions is a two-dimensional vector space.
First recall that we have to superposition principle, i.e. linear combinations of solutions are still solutions, so the solution space is a vector space. Second we have to specify two initial conditions, $x(0)$ and $x'(0)$, so the dimension of this vector space is at least 2. On the other hand, an initial value problem, setting $x(0)$ and $x'(0)$, has a unique solution, which means the dimension of the vector space of solutions is precisely 2.

(b) Prove that if x_1 and x_2 are solutions and they are linearly independent at $t = 0$ (i.e. one can write the solution to any initial value problem with $x(0) = a$ and $x'(0) = b$ as a linear combination of x_1 and x_2) then x_1 and x_2 are linearly independent for all t.

6. Consider the differential equation

$$x'' - 6x' + 9x = \sin t.$$

(a) Find the general solution to the associated homogeneous problem.
The homogeneous equation is $x'' - 6x' + 9x = 0$. We try solutions of the form $x(t) = e^{rt}$, so then

$$0 = r^2 - 6r + 9 = (r - 3)^2 \Rightarrow r = 3.$$

Observe this is a double root, so the general solution is

$$x(t) = c_1 e^{3t} + c_2 t e^{3t}.$$

(b) Find a solution to the inhomogeneous problem using your favorite method.
We try to find a solution of the form

$$x_p(t) = A \sin t + B \cos t.$$

Plugging in, we have

$$\begin{aligned} \sin t &= x_p'' - 6x_p' + 9x_p = -(A \sin t + B \cos t) - 6(A \cos t - B \sin t) + 9(A \sin t + B \cos t) \\ &= \sin t(-A + 6B + 9A) + \cos t(-B - 6A + 9B). \end{aligned}$$

This gives us two equations in A and B (matching the $\sin t$ and $\cos t$ terms):

$$8A + 6B = 1, \qquad -6A + 8B = 0,$$

which has the solution $A = 2/25$, $B = 3/50$, so

$$x_p = \frac{2}{25}\sin t + \frac{3}{50}\cos t.$$

(c) Find the solution to the inhomogeneous initial value problem with $x(0) = 1$ and $x'(0) = -1$.
The general solution to the homogeneous equation is the sum of the homogeneous solution together with the particular solution we just found:

$$x(t) = \frac{2}{25}\sin t + \frac{3}{50}\cos t + c_1 e^{3t} + c_2 t e^{3t}.$$

We need to find the values of c_1 and c_2 which match the given initial conditions:

$$1 = x(0) = \frac{3}{50} + c_1, \qquad -1 = x'(0) = \frac{2}{25} + 3c_1 + c_2.$$

Solving this system of equations for c_1 and c_2 we get $c_1 = -3/50$ and $c_2 = -9/10$, so

$$x(t) = \frac{2}{25}\sin t + \frac{3}{50}\cos t - \frac{3}{50}e^{3t} - \frac{9}{10}te^{3t}.$$

7. Consider the differential equation

$$x'' - 2x' + x = e^t.$$

(a) Find the general solution to the associated homogeneous problem.
The associated homogeneous equation is $x'' - 2x + x = 0$, and we again look for equations of the form $x = e^{rt}$. Then

$$0 = x'' - 2x' + x = e^{rt}(r^2 - 2r + 1) = e^{rt}(r-1)^2 \Rightarrow r = 1.$$

Notice this time we have a double root, so the general solution is

$$x = c_1 e^t + c_2 t e^t.$$

(b) Find a solution to the inhomogeneous problem using your favorite method.
First of all, the right hand side is a solution to the homogeneous equation. Also, the homogeneous equation already has a double root. This means we should look for a solution of the form $x_p = At^2 e^t$. Plugging this in we get

$$\begin{aligned} e^t &= x_p'' - x_p' + x_p \\ &= (2Ae^t + 4Ate^t + Ae^{t^2}) - 2(2Ate^t + At^2e^t) + At^2e^t \\ &= 2Ae^t \Rightarrow A = 1/2, \end{aligned}$$

and so a particular solution is $x_p = (1/2)t^2 e^t$.

(c) Find the solution to the inhomogeneous initial value problem with $x(0) = 2$ and $x'(0) = 0$.
The general solution to the nonhomogeneous equation is

$$x = \frac{t^2}{2}e^t + c_1 e^t + c_2 t e^t.$$

Now we plug in the initial conditions to find c_1 and c_2:

$$2 = x(0) = c_1, \qquad 0 = x'(0) = c_1 + c_2 \Rightarrow c_2 = -2,$$

and so

$$x(t) = e^t(2 - 2t + t^2/2).$$

8. Consider the system of differential equations

$$u' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} u.$$

(a) Find the general solution using your favorite method.

We will find solutions by looking for eigenvalues and eigenvectors. First we find eigenvalues:

$$0 = \det \begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} = \lambda^2 - 5\lambda + 6 - 2 = (\lambda - 1)(\lambda - 4) \Rightarrow \lambda = 1, 4.$$

Next we find the $\lambda = 1$ eigenvector:

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

This gives the equation $3a + b = a \Rightarrow b = -2a$, so we can take the eigenvector to be

$$\vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Next we find the $\lambda = 4$ eigenvector:

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4a \\ 4b \end{bmatrix}.$$

This gives the equation $3a + b = 4a \Rightarrow b = a$, so we can take the eigenvector to be

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Finally, putting everything together, we have the general solution

$$u(t) = c_1 e^t \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Find the solution to the initial value problem with $u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We plug in the initial conditions in to solve for c_1 and c_2 in the general solution:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow c_1 = 0, c_2 = 1.$$

Thus the solution to the initial value problem is

$$u(t) = e^{4t} \begin{bmatrix} 1 \end{bmatrix}.$$

(c) What is the stability behavior of the fixed point $u = 0$? Explain your answer.

Notice that both eigenvalues are positive, so the origin is unstable. In fact, it's a source.

9. Consider the system

$$u' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} u.$$

(a) Find the general solution using your favorite method.

We will find solutions using eigenvalues and eigenvectors again. First we find eigenvalues:

$$0 = \det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 + 1 = (\lambda - 2)^2 \Rightarrow \lambda = 2.$$

So we have a double root for the eigenvalue. Now look for eigenvectors:

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix} \Rightarrow a = b.$$

This means that (up to scale) the only eigenvector is

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so $\lambda = 2$ is a deffective eigenvalue. This means the general solution is

$$\vec{u}(t) = e^{2t}\vec{u}(0) + c_1 te^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In fact, if we let

$$\vec{u}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

then we see

$$\vec{u}(t) = e^{2t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{2t}\begin{bmatrix} x_0 - y_0 \\ x_0 - y_0 \end{bmatrix}.$$

(b) Solve the initial value problem with $u(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We've already done all the work in the last part. Plug this initial condition in to see

$$\vec{u}(t) = e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

(c) Classify the origin as stable or unstable.

The eigenvalues are positive, so solutions all grow exponentially, which means the origin is unstable.

10. Consider a buoy suspended in water, shaped like a cylinder or radius r and height h. Let the buoy have uniform density $\rho \leq 1/2$ and remember that water has density 1. Let $x = x(t)$ be the depth to which the buoy is submerged and suppose $x(0) = 0$.

(a) The forces acting on the buoy are a gravitational force of mg and a buoyancy force of $\pi r^2 gx$. Use this information to write down a differential equation for x.

We start with $F = ma$, where F is the force, m is the mass, and a is the acceleration. Because the buoy is a cylinder with uniform density ρ, the mass is $m = \rho\pi r^2 h$. The force is a combination of the gravitational force, $mg = \rho\pi r^2 hg$, and the buoancy force $-\pi r^2 gx$. (Recall that x is the amount the buoy is submerged, so x increases as the buoy moves down.) Putting this together we have

$$\rho\pi r^2 hx'' = \rho\pi r^2 hg - \pi r^2 gx \Rightarrow x'' + \frac{g}{h\rho}x = g.$$

(b) Conclude that the buoy undergoes simple harmonic motion with equilibrium $x_0 = \rho h$ and period $2\pi\sqrt{\rho h/g}$.

Let's look at our differential equation. It's second order, linear, and nonhomogeneous. The homogeneous equation is $x'' + (g/h\rho)x = 0$, and has oscillating solutions $c_1 \cos(\sqrt{g/h\rho}t) + c_2 \sin(\sqrt{g/h\rho}t)$. Now we have to find a particular solution to the inhomogeneous equation. Because the right hand side is a constant, we try $x_p = A$. Then

$$g = x_p'' + \frac{g}{h\rho}x_p = \frac{gA}{h\rho} \rightarrow A = h\rho.$$

Thus the general solution to the inhomogeneous equation is

$$x(t) = h\rho + c_1 \cos(\sqrt{g/h\rho}t) + c_2 \sin(\sqrt{g/h\rho}t),$$

which is simple harmonic motion translated by $h\rho$. In other words, the equilibrium value for x is $h\rho$. Also, the period is

$$T = \frac{2\pi}{\sqrt{g/(h\rho)}} = 2\pi\sqrt{\frac{h\rho}{g}}.$$

(c) Compute the period and amplitude of the motion when $\rho = .5\text{g/cm}^3$, $h = 200\text{cm}$, and $g = 980\text{cm/s}^2$.

Now we plug in some actual numbers:

$$x(t) = 100 + c_1 \cos(\sqrt{98}t) + c_2 \sin(\sqrt{98}t).$$

The initial conditions are $x(0) = 0 = x'(0)$, so solving for the constants we get $c_1 = -100$ and $c_2 = 0$, which means

$$x(t) = 100(1 - \cos(\sqrt{98}t)).$$

Thus the amplitude is 100 and the priod is $2\pi/\sqrt{98} \simeq .635$.

11. Consider the soft spring equation

$$x'' + 4x - x^3 = 0.$$

(a) Let $v = x'$ and rewrite the second order scalar equation as a first order system in x and v.

If we let $x' = v$, this system becomes

$$x' = v, \quad v' = x'' = -4x + x^3 \Leftrightarrow \begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} v \\ -4x + x^3 \end{bmatrix}.$$

(b) Show that the quantity

$$E = \frac{v^2}{2} + 2x^2 - \frac{x^4}{4}$$

is conserved by solutions. (Hint: take the derivative of E with respect to t). Conclude that solutions must lie on curves in the (x, v) plane where E is constant.

We take a derivative of the energy:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}(v^2/2 + 2x^2 - x^4/4) \\ &= vv' + 4xx' - x^3x' \\ &= x'x'' + 4xx' - x^3x' \\ &= x'(x'' + 4x - x^3) = 0. \end{aligned}$$

Because the energy E is constant, for any solution curve we must have $v^2/x + 2x^2 - x^4/4$ constant, which is the same thing as saying solution curves lie on level sets of E.

(c) Show that the only equilibria are $(x, v) = (0, 0)$, $(x, v) = (2, 0)$ and $(x, v) = (-2, 0)$.

The equilibria satisfy $x' = 0 = v'$, which we can rewrite (using the system) as

$$v = 0, \qquad -4x + x^3 = 0.$$

We can factor the cubic in the second equation:

$$0 = x^3 - 4x = x(x-2)(x+2) \Rightarrow x = 0, \pm 2.$$

This implies all our fixed points are $(0, 0)$, $(2, 0)$, and $(-2, 0)$.

(d) Linearize the system about each equilibrium and classify the point as stable or unstable.

If we let $g(x, v) = v$ and $h(x, v) = -4x + x^3$, then the system is

$$\begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} g(x,v) \\ h(x,v) \end{bmatrix} = \begin{bmatrix} v \\ -4x + x^3 \end{bmatrix}.$$

We linearize about a fixed point by taking derivatives, so we need to know

$$\frac{\partial g}{\partial x} = 0 = \frac{\partial h}{\partial y}, \quad \frac{\partial g}{\partial v} = 1, \quad \frac{\partial h}{\partial x} = -4 + 3x^2.$$

At the origin, the matrix of the linearization is

$$J = \begin{bmatrix} \partial g/\partial x & \partial g/\partial y \\ \partial h/\partial x & \partial h/partialy \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

This matrix has eigenvalues λ where

$$0 = \det \begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix} = \lambda^2 + 4 \rightarrow \lambda = \pm 4i.$$

This means we have to do some more analysis to see the stability of $(0, 0)$.

At the $(\pm 2, 0)$, the matrix of the linearization is

$$J = \begin{bmatrix} \partial g/\partial x & \partial g/\partial y \\ \partial h/\partial x & \partial h/partialy \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}.$$

This matrix has eigenvalues λ where

$$0 = \det \begin{bmatrix} -\lambda & 1 \\ 8 & -\lambda \end{bmatrix} = \lambda^2 - 8 \to \lambda = \pm 2\sqrt{2}.$$

This means $(\pm 2, 0)$ are both saddle points, and so in particular they're unstable.

Now we can see about the stability of $(0,0)$. Recall that the solution curves lie on level sets of $E = v^2/2 + 2x^2 - x^4/4$. The $E = 4$ level set passes through both equilibria $(\pm 2, 0)$, and in particular it surrounds the origin. Because solution curves can't cross, all the solution curves which start near $(0,0)$ have to stay inside the $E = 4$ level set, which means all these curves loop around the origin. This means $(0,0)$ is a stable center.

(e) Sketch some representative solution curves.

12. Consider the one-parameter family of differential equations with parameter α given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_1 - \alpha x_1 x_2 \\ x_2 + \alpha x_1 x_2 \end{pmatrix} = F(x_1, x_2).$$

(a) Verify that the only fixed points of this system are $(0,0)$ and $(-1/\alpha, 1/\alpha)$ for $\alpha \neq 0$, and only $(0,0)$ for $\alpha = 0$.

Fixed points occur when $F(x_1, x_2) = 0$, which is equivalent to

$$0 = x_1(1 - \alpha x_2), \qquad 0 = x_2(1 + \alpha x_1).$$

This occurs precisely at

$$(0,0), \qquad (-1/\alpha, 1/\alpha).$$

(b) Linearize this system about each fixed point.

The coefficient matrix of the linearization is given by the derivative of F, which is

$$DF = \begin{bmatrix} 1 - \alpha x_2 & -\alpha x_1 \\ \alpha x_2 & 1 + \alpha x_1 \end{bmatrix}.$$

Evaulating this at $(0,0)$, we have

$$DF(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Evaulating this at $(-1/\alpha, 1/\alpha)$, we have

$$DF(-1/\alpha, 1/\alpha) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(c) What is the stability behavior of each fixed point? Explain your answer.

The eigenvalues of $DF(0,0)$ are both 1, so this fixed point is unstable. The eigenvalues of $DF(-1/\alpha, 1/\alpha)$ are ± 1, so this fixed point is also unstable (one positive eigenvalue).

(d) Draw some representative phase portraits.

(e) Is there a bifurcation point in α? If there is, find it. Explain your answer.

Yes; there is only one fixed point for $\alpha = 0$, while there are two fixed points for $\alpha \neq 0$.

13. Consider the one-parameter family of differential equations with parameter α given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} (e^{x_1} - 1)(\alpha - x_2) \\ (e^{x_2} - 1)(\alpha - x_1) \end{pmatrix} = F(x_1, x_2).$$

(a) Verify that the only fixed points of this system are $(0,0)$ and (α, α).

The fized points occur when

$$0 = (e^{x_1} - 1)(\alpha - x_2), \qquad 0 = (e^{x_2} - 1)(\alpha - x_1),$$

and this happens precisely when $(x_1, x_2) = (0,0)$ or $(x_1, x_2) = (\alpha, \alpha)$.

(b) Linearize this system about each fixed point.

The coefficient matrix of the linearization is given by

$$DF = \begin{bmatrix} (\alpha - x_2)e_1^x & 1 - e^{x_1} \\ 1 - e^{x_2} & (\alpha - x_1)e^{x_2} \end{bmatrix}.$$

Evaluating at $(0,0)$, we have

$$DF(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix},$$

Evaluating at (α,α), we have

$$DF(\alpha,\alpha) = \begin{bmatrix} 0 & 1 - e^{\alpha} \\ 1 - e^{\alpha} & 0 \end{bmatrix}.$$

(c) What is the stability behavior of each fixed point? Explain your answer.

The eigenvalues of $DF(\alpha,\alpha)$ are $\pm(1 - e^{\alpha})$, which assumes both signs. Thus (α,α) is always unstable. The eigenvalues of $DF(0,0)$ are both α, which means that the system is stable if and only if $\alpha < 0$.

(d) Draw some representative phase portraits.

(e) Is there a bifurcation point in α? If there is, find it. Explain your answer.

Yes, the stability behavior of the fixed point $(0,0)$ changes as α passes from positive to negative.

14. Compute the Laplace transform of the function $f(t) = te^t$.

$$\begin{aligned} \mathcal{L}(f)(s) &= \int_0^\infty e^{-st}te^t dt = \int_0^\infty te^{(1-s)t}dt \\ &= \frac{t}{1-s}e^{(1-s)t}\Big|_{t=0}^{t=\infty} - \frac{1}{1-s}\int_0^\infty e^{(1-s)t}dt \\ &= -\frac{1}{(1-s)^2}e^{(1-s)t}\Big|_{t=0}^{t=\infty} = \frac{1}{(1-s)^2} \end{aligned}$$

15. (a) If $g(t) = f(-t)$, show that $\mathcal{L}(g)(s) = -\mathcal{L}(f)(-s)$.

Implicit in here is that we have to reverse the sign of s, changing s to $-s$ when we change variables. The change of variables is $\tau = -t$, by the way.

$$\mathcal{L}(g)(s) = \int_0^\infty e^{-st}g(t)dt = \int_0^\infty e^{-st}f(-t)dt = -\int_0^\infty e^{st}f(t)dt = -\int_0^\infty e^{-(-s)t}f(t)dt = -\mathcal{L}(f)(-s).$$

(b) If $g(t) = f(at)$ (where $a > 0$ is a constant), show that $\mathcal{L}(g)(s) = \frac{1}{a}\mathcal{L}(f)(s/a)$.

This time the change of variables is $\tau = at$

$$\mathcal{L}(g)(s) = \int_0^\infty e^{-st}g(t)dt = \int_0^\infty e^{-st}f(at)dt = \frac{1}{a}\int_0^\infty e^{s\tau/a}f(\tau)d\tau = \frac{1}{a}\mathcal{L}(f)(s/a).$$

16. Recall that for a function f to be of exponential order it means that $\lim_{t\to\infty} e^{-ct}|f(t)| = 0$ for some choice of c.

(a) Show that $f(t) = \cos(e^{t^2})$ is of exponential order. (In fact, it is bounded, which is much better than exponential order.)

$$|f(t)| = |\cos(e^{t^2})| \leq 1.$$

(b) Show that f' is not of exponential order.

First we compute $f'(t) = -2te^{t^2}\sin(e^{t^2})$. Now the derivative grows like $2te^{t^2}$, which is faster than exponential growth. We can see that by evaluating f' at $t_n = \sqrt{\ln((2n+1)\pi/2)}$.

17. (a) Solve the initial value problem

$$x'' + tx' + x = 0 \quad x(0) = 1 \quad x'(0) = 0.$$

(Hint: try taking the Laplace transform of both sides.)

Taking the Laplace transform of both sides we see

$$\begin{aligned} 0 &= \mathcal{L}(x'') + \mathcal{L}(tx') + \mathcal{L}(x) = s\mathcal{L}(x') - x'(0) + \frac{d}{ds}(\mathcal{L}(x')) + \mathcal{L}(x) \\ &= s^2\mathcal{L}(x) - sx(0) - x'(0) + \frac{d}{ds}(s\mathcal{L}(x) - x(0)) + \mathcal{L}(x) = s^2\mathcal{L}(x) - s + 2\mathcal{L}(x) + s\frac{d}{ds}\mathcal{L}(x). \end{aligned}$$

Rearranging this equation we see

$$\frac{d}{ds}\mathcal{L}(x) + (2 + s^2)\mathcal{L}(x) = s,$$

which is a linear first order equation in $\mathcal{L}(x)$. This first order equation has solutions

$$\mathcal{L}(x)(s) = e^{-\frac{1}{3}s^3 - 2s}(\int_0^s \tau e^{\frac{1}{3}\tau^3 + \tau} d\tau + c).$$

We can now recover x by computing the inverse Laplace transform.

(b) Solve the initial value problem

$$x'' + e^t x' - x = 0 \quad x(0) = -1 \quad X'(0) = 1.$$

(Hint: try taking the Laplace transform of both sides.)

Taking the Laplace transform of both sides we see

$$\begin{aligned} 0 &= \mathcal{L}(x'') + \mathcal{L}(e^t x') - \mathcal{L}(x) = s^2\mathcal{L}(x)(s) - sx(0) - x'(0) + \mathcal{L}(x')(s-1) - \mathcal{L}(x)(s) \\ &= s^2\mathcal{L}(x)(s) + s - 1 + \mathcal{L}(x)(s-1) - 1 - \mathcal{L}(x). \end{aligned}$$

Rearranging this we see

$$2 - s = (s^2 - 1)\mathcal{L}(x)(s) + \mathcal{L}(x)(s-1).$$

Solving for $\mathcal{L}(x)$ we get

$$\mathcal{L}(x)(s) = \frac{2-s}{s^2+s-2} = \frac{2}{(s+2)(s-1)} - \frac{s}{(s+2)(s-1)} = \frac{-2/3}{s+2} + \frac{2/3}{s-1} + \frac{-2/3}{s+2} + \frac{-1/3}{s-1} = \frac{-4/3}{s+2} + \frac{1/3}{s-1}.$$

Now take the inverse Laplace transform:

$$x = \mathcal{L}^{-1}\left(\frac{-4/3}{s+2} + \frac{1/3}{s-1}\right) = -\frac{4}{3}e^{-2t} + \frac{1}{3}e^t.$$