Practice Problems

Math 221

March 21, 2007

1. Find the general solution of each of the following differential equations.

   (a) $u'' - 2u' + u = 0$

   (b) $u'' - 3u' + 2u = x$

   (c) $u'' - 3u' + 2u = e^x$

   (d) $u'' - 2u' + u = e^x$

   (e) $u'' - 5u' + 6u = e^x + \sin x$

2. Solve the following initial value problems.

   (a) $u'' - 3u' + 2u = x$, $u(0) = -1$, $u'(0) = 1$

   (b) $u'' - 3u' + 2u = e^x$, $u(0) = 0$, $u'(0) = 0$

   (c) $u'' + 3u' + 2u = \sin x$, $u(0) = 1$, $u'(0) = -2$

3. Consider a mass-spring system with mass m, spring constant k, damping constant c, initial position $x(0) = x_0$, and initial velocity $x'(0) = v_0$. Let $p = c/(2m)$, $\omega_-^2 = k/m$, and $\omega_1^2 = \omega_0^2 - p^2$. This problem will consider the critically damped (i.e. $c^2 = 4km$) case.

   (a) Show that
   $$x(t) = (x_0 + v_0 t + p x_0 t)e^{-pt}.$$

   (b) Show that $x(T) = 0$ for some $T > 0$ if and only if x_0 and $v_0 + x_0 p$ have opposite signs.

   (c) Conclude that $x(t)$ has a local maximum or minimum at some positive T if and only if x_0 and $v_0 + x_0 p$ have the same sign.

4. Consider a mass-spring system with mass m, spring constant k, damping constant c, initial position $x(0) = x_0$, and initial velocity $x'(0) = v_0$. Let $p = c/(2m)$, $\omega_-^2 = k/m$, and $\omega_1^2 = \omega_0^2 - p^2$. This problem will consider the underdamped (i.e. $c^2 < 4km$) case.

   (a) Show that
   $$x(t) = e^{-pt}(x_0 \cos(\omega_1 t) + \frac{v_0 + p x_0}{\omega_1} \sin(\omega_1 t)).$$

   (b) Suppose that c is much smalller than $\sqrt{8mk}$. In this case, use the binomial series to show
   $$\omega_1 \simeq \omega_0(1 - \frac{c^2}{8mk}).$$
   The binomial series is
   $$(a+b)^n = \sum_{j=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}.$$

   (c) Now for a specific problem. Suppose a body weighs 100 lb (so the mass is $m = 3.125$ slugs, in fps units) is oscillating atached to a spring and a dashpot. Its first two maximum displacements are $x = 6.73$ in, occuring at $t = .34$ s, and $x = 1.46$ in, occuring at $t = 1.17$ s. Compute the damping constant c and the spring constant k.

5. Explain why the method of undetermined coefficients will not work for the differential equation
   $$u'' + 3u' + 2u = \frac{1}{1+x^2}.$$
   What if the right hand side is xe^{x^2}? Can you use the method of undetermined coefficients for that right hand side?

6. **Without solving** the following systems of differential equations, determine whether or not the origin ($u_1 = 0 = u_2$) is unstable, stable, or strictly stable.

(a) $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

(b) $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

(c) $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

(d) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

7. Find the general solution of each of the following systems.

(a) $\begin{aligned} u_1 + u_2 &= u_1' \\ -u_2 &= u_2' \end{aligned}$

(b) $\begin{aligned} u_1 &= u_2' \\ u_2 &= -2u_1' \end{aligned}$

(c) $\begin{aligned} u_1 + 2u_2 &= u_1' \\ 2u_1 + u_2 &= u_2' \end{aligned}$

(d) $\begin{aligned} u_1 - u_2 &= u_1' \\ u_1 + u_2 &= u_2' \end{aligned}$