Solutions to the Practice Problems

Math 221

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1. Find the general solution of each of the following differential equations.

   (a) $u'' - 2u' + u = 0$

   We try solutions of the form $u = e^{rx}$, then

   $$0 = r^2 - 2r + 1 = (r-1)^2 \Rightarrow r = 1.$$

   Notice in this case we have a double root, so the general solution is

   $$u(x) = (c_1 + c_2 x)e^x.$$

   (b) $u'' - 3u' + 2u = x$

   We first find a particular solution to the inhomogeneous equation. Notice the right hand side is a first degree polynomial, so we try another first degree polynomial:

   $$u_p = Ax + B \Rightarrow x = u_p'' - 3u_p' + 2u_p = -3A + 2(Ax + B).$$

   Matching coefficients, we see $2A = 1$ and $2B - 3A = 0$. The first equation tells us $A = 1/2$, and the second equation tells us $B = 3/4$.

   Next we find the general solution to the homogeneous equation $u'' - 3u' + 2u = 0$. We try a solution of the form $u = e^{rx}$, which forces

   $$0 = r^2 - 3r + 2 = (r-2)(r-1) \Rightarrow r = 1, 2.$$

   So the general solution of the homogeneous equation is $c_1 e^x + c_2 e^{2x}$, which means the general solution to the inhomogeneous equation is

   $$u = \frac{1}{2}x + \frac{3}{4} + c_1 e^x + c_2 e^{2x}.$$

   (c) $u'' - 3u' + 2u = e^x$

   We can reuse the homogeneous solution we already found, but this time the right hand side is a solution to the homogeneous equation. So we try a particular solution of the form

   $$u_p = Axe^x \Rightarrow u_p' = Axe^x + Ae^x, \qquad u_p'' = Axe^x + 2Ae^x,$$

   so

   $$e^x = u_p'' - 3u_p' + 2u_p = e^x[(A - 3A + 2A)x + (2A - 3A)] = -Ae^x \Rightarrow A = -1.$$

   Thus a particular solution is $u_p = -xe^x$ and the general solution is

   $$u = -xe^x + c_1 e^x + c_2 e^{2x}.$$

   (d) $u'' - 2u' + u = e^x$

   We can reuse the general solution to the homogeneous equation $(c_1 + c_2 x)e^x$ we found previously, but we still have to find a particular solution to the inhomogeneous equation. We try a solution of the form

   $$u_p = Ax^2 e^x \Rightarrow u_p' = Ax^2 e^x + 2Axe^x, \qquad u_p'' = Ax^2 e^x + 4Axe^x + 2Ae^x,$$

   so then

   $$e^x = u_p'' - 2u_p' + u_p = e^x[Ax^2 + 4Ax + 2A - 2Ax^2 - 4Ax + Ax^2] = 2Ae^x \Rightarrow A = \frac{1}{2}.$$

   Thus a particular solution is $u_p = (1/2)x^2 e^x$ and the general solution is

   $$u = \frac{1}{2}x^2 e^x + (c_1 + c_2 x)e^x.$$

(e) $u'' - 5u' + 6u = e^x + \sin x$

We first solve the associated homogeneous problem: $u'' - 5u' + 6u = 0$. Looking for solutions of the form $u = e^{rx}$ we find

$$0 = r^2 - 5r + 6 = (r-2)(r-3) \Rightarrow r = 2, 3,$$

and so the homogeneous equation has solutions of the form $c_1 e^{2x} + c_2 e^{3x}$. Now we find a particular solution to the homogeneous equation. Because the right hand side is a mix of trig and exponential functions, we try

$$u_p = Ae^x + B\sin x + C\cos x \Rightarrow u_p' = Ae^x + B\cos x - C\sin x, \qquad u_p'' = Ae^x - B\sin x - C\cos x,$$

and so

$$e^x + \sin x = u_p'' - 5u_p' + 6u_p = Ae^x - B\sin x - C\cos x - 5Ae^x - 5B\cos x + 5C\sin x + 6Ae^x + 6B\sin x + 6C\cos x.$$

Matching the exponential terms, we get $A = 1/2$. Matching the trig terms, we get two equations

$$1 = 5B + 5C, \qquad 0 = -5B + 5C \Rightarrow B = \frac{1}{10} = C.$$

Putting everything together, we have a general solution of

$$u = \frac{1}{2}e^x + \frac{1}{10}\sin x + \frac{1}{10}\cos x + c_1 e^{2x} + c_2 e^{3x}.$$

2. Solve the following initial value problems.

(a) $u'' - 3u' + 2u = x$, $u(0) = -1$, $u'(0) = 1$

Here we're able to use the general solution we found above:

$$u = \frac{1}{2}x + \frac{3}{4} + c_1 e^x + c_2 e^{2x}.$$

Evaluate at the initial point $x = 0$:

$$-1 = u(0) = \frac{3}{4} + c_1 + c_2, \qquad 1 = u'(0) = \frac{1}{2} + c_1 + 2c_2.$$

This is two equations in the two unknowns c_1 and c_2, which we can solve by taking the difference of the equations. This gives us $c_2 = 9/4$ and $c_1 = -4$, so the solution is

$$u = \frac{1}{2}x + \frac{3}{4} - 4e^x + \frac{9}{4}e^{2x}.$$

(b) $u'' - 3u' + 2u = e^x$, $u(0) = 0$, $u'(0) = 0$

Again, we can use the general solution we found above,

$$u = -xe^x + c_1 e^x + c_2 e^{2x}.$$

Evaluate at the initial point $x_0 = 0$:

$$0 = u(0) = c_1 + c_2, \qquad 0 = u'(0) = -1 + c_1 + 2c_2.$$

The solution of this 2×2 system is $c_2 = 1$ and $c_1 = -1$, so

$$u(x) = -xe^x - e^x + e^{2x}.$$

(c) $u'' + 3u' + 2u = \sin x$, $u(0) = 1$, $u'(0) = -2$

This time we need to find the general solution. The associated homogeneous solution is $u'' + 3u' + 2u = 0$, and has the solution $u = c_1 e^{-x} + c_2 e^{-2x}$. To find a particular solution, we try

$$u_p = A\sin x + B\cos x \Rightarrow u_p' = A\cos x - B\sin x, \qquad u_p'' = -A\sin x - B\cos x,$$

and so

$$\sin x = u_p'' + 3u_p' + 2u_p = -A\sin x - B\cos x + 3A\cos x - 3B\sin x + 2A\sin x + 2B\cos x.$$

Matching coefficients we see $A = 1/10$ and $B = -3/10$, so the general solution is

$$u = \frac{1}{10}\sin x - \frac{3}{10}\cos x + c_1 e^{-x} + c_2 e^{-2x}.$$

Evaluate at the initial point $x = 0$:

$$1 = u(0) = -\frac{3}{10} + c_1 + c_2, \qquad -2 = \frac{1}{10} - c_1 - 2c_2,$$

which has the solution $c_1 = 1/2$ and $c_2 = 4/5$. Thus the solution is

$$u = \frac{1}{10}\sin x - \frac{3}{10}\cos x + \frac{1}{2}e^{-x} + \frac{4}{5}e^{-2x}.$$

3. Consider a mass-spring system with mass m, spring constant k, damping constant c, initial position $x(0) = x_0$, and initial velocity $x'(0) = v_0$. Let $p = c/(2m)$, $\omega_{-}^2 = k/m$, and $\omega_1^2 = \omega_0^2 - p^2$. This problem will consider the critically damped (i.e. $c^2 = 4km$) case.

   (a) Show that

   $$x(t) = (x_0 + v_0 t + p x_0 t)e^{-pt}.$$

   The differential equation is

   $$mx'' + cx' + kx = 0,$$

   and we look for solutions of the form $x(t) = e^{rt}$. Plugging this in, we see

   $$mr^2 + cr + k = 0 \Rightarrow r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} = -p.$$

   Here we have used the fact that $c^2 = 4km$. Because we have a double root, the general solution is

   $$x(t) = (c_1 + c_2 t)e^{-pt}.$$

   Now we match the initial conditions:

   $$x_0 = x(0) = c_1, \qquad v_0 = x'(0) = -pc_1 + c_2 = -px_0 + c_2 \Rightarrow c_2 = v_0 + px_0.$$

   Putting this all together we get

   $$x(t) = (x_0 + t(v_0 + px_0))e^{-pt}$$

   as claimed.

   (b) Show that $x(T) = 0$ for some $T > 0$ if and only if x_0 and $v_0 + x_0 p$ have opposite signs.

   Now suppose $x(T) = 0$ for some $T > 0$. This means

   $$0 = x(T) = (x_0 + T(v_0 + px_0))e^{-pT} \Rightarrow x_0 = -T(v_0 + px_0).$$

   Because $T > 0$, this is only possible if x_0 and $v_0 + px_0$ have opposite signs.

   (c) Conclude that $x(t)$ has a local maximum or minimum at some positive T if and only if v_0 and $v_0 + x_0 p$ have the same sign.

   Now let x have a critical point at some $T > 0$. Then

   $$0 = x'(T) = -px_0 e^{-pT} + (v_0 + px_0)e^{-pT} - p(v_0 + px_0)Te^{-pT} = (v_0 - pT(v_0 + px_0))e^{-pT},$$

   which in turn implies

   $$v_0 = pT(v_0 + px_0).$$

   Because $pT > 0$ this means v_0 and $v_0 + px_0$ have the same sign. This is what we want to show.

4. Consider a mass-spring system with mass m, spring constant k, damping constant c, initial position $x(0) = x_0$, and initial velocity $x'(0) = v_0$. Let $p = c/(2m)$, $\omega_{-}^2 = k/m$, and $\omega_1^2 = \omega_0^2 - p^2$. This problem will consider the underdamped (i.e. $c^2 < 4km$) case.

(a) Show that

$$x(t) = e^{-pt}(x_0\cos(\omega_1 t) + \frac{v_0 + px_0}{\omega_1}\sin(\omega_1 t)).$$

The differential equation is

$$mx'' + cx' + kx = 0.$$

We try solution of the form $x(t) = e^{rt}$, so $mr^2 + cr + k = 0$, which has the solutions

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c}{2m} \pm i\sqrt{\frac{4km}{4m^2} - \frac{c^2}{4m^2}} = -p \pm i\sqrt{k/m - p^2} = -p \pm i\sqrt{\omega_0^2 - p^2}.$$

If we let $\omega_1 = \sqrt{\omega_0^2 - p^2}$ then we see that the solution has the form

$$x(t) = e^{-pt}(c_1\cos\omega_1 t + c_2\sin\omega_1 t).$$

Now match the initial conditions:

$$x_0 = x(0) = c_1, \qquad v_0 = x'(0) = -pc_1 + c_2\omega_1 \Rightarrow c_2 = \frac{v_0 + px_0}{\omega_1}.$$

Putting this all together, we get

$$x(t) = e^{-pt}(x_0\cos(\omega_1 t) + (\frac{v_0 + px_0}{\omega_1})\sin(\omega_1 t))$$

as claimed.

(b) Suppose that c is much smalller than $\sqrt{8mk}$. In this case, use the binomial series to show

$$\omega_1 \simeq \omega_0(1 - \frac{c^2}{8mk}).$$

The binomial series is

$$(a+b)^n = \sum_{j=0}^{n} \frac{n!}{k!(n-k)!}a^k b^{n-k}.$$

The new frequency is

$$\omega_1 = \sqrt{\omega_0^2 - p^2} = \omega_0\sqrt{1 - \frac{p^2}{\omega_0^2}} = \omega_0\sqrt{1 - \frac{c^2}{4m^2}\cdot\frac{k}{m}} = \omega_0\sqrt{1 - \frac{c^2}{4km}}.$$

Now we suppose $c^2/(4km)$ is very small, and use that fact that for x small $\sqrt{1-x} \simeq 1 - x/2$ (this is essentially tangent line approximation):

$$\omega_1 = \omega_0\sqrt{1 - \frac{c^2}{4km}} \simeq \omega_0(1 - \frac{c^2}{8km}).$$

(c) Now for a specific problem. Suppose a body weighs 100 lb (so the mass is $m = 3.125$ slugs, in fps units) is oscillating atached to a spring and a dashpot. Its first two maximum displacements are $x = 6.73$ in, occuring at $t = .34$ s, and $x = 1.46$ in, occuring at $t = 1.17$ s. Compute the damping constant c and the spring constant k.

First we take the derivative of

$$x(t) = e^{-pt}(x_0\cos(\omega_1 t) + (\frac{v_0 + x_0 p}{\omega_1})\sin(\omega_1 t))$$

to find critical points:

$$0 = x'(t) = e^{-pt}\left[v_0\cos(\omega_1 t) - (\frac{v_0 p + x_0\omega_1 p + x_0 p^2}{\omega_1})\sin(\omega_1 t)\right] \Rightarrow \tan(\omega_1 t) = \frac{v_0\omega_1}{v_0 P + x_0\omega_1 p + x_0 p^2}.$$

Notice that $\tan(\omega_1 t)$ is $2\pi/\omega_1$–periodic, and successive maxima occur every other critical point. Thus

$$\frac{2\pi}{\omega_1} = t_2 - t_2 = 1.17 - .34 = .83 \Rightarrow \omega_1 = \frac{2\pi}{.34} \simeq 7.57.$$

Now we evaluate x at successive maxima. Because the sine and cosine terms are both $2\pi/\omega_1$–periodic, we get

$$.217 \simeq \frac{1.46}{6.73} = \frac{x_2}{x_1} = \frac{x(t_1 + 2\pi/\omega_1)}{x(t_1)} = \frac{e^{-pt_1 - 2\pi p/\omega_1}}{e^{-pt_1}} = e^{-2\pi p/\omega_1}.$$

Now we solve for vaarious things.

$$\frac{c}{2m} = p = -\frac{\omega_1}{2\pi}\ln(.217) \simeq 1.84 \Rightarrow c \simeq 11.5.$$

Also,

$$\frac{k}{m} = \omega_0^2 = \omega_1^2 + p^2 \simeq (7.57)^2 + (1.84)^2 \simeq 60.7 \Rightarrow \omega_0 \simeq 7.79 \Rightarrow k \simeq 24.3.$$

5. Explain why the method of undetermined coefficients will not work for the differential equation

$$u'' + 3u' + 2u = \frac{1}{1+x^2}.$$

What if the right hand side is xe^{x^2}? Can you use the method of undetermined coefficients for that right hand side?

First consider the right hand side $(1+x^2)^{-1}$. Notice that when you take derivatives, of this right hand side, you get larger (negative) powers of x, so the derivatives never repeat or terminate. This means you can't guess what combination of derivatives of $(1+x^2)^{-1}$ (and similar functions) will combine to cancel and produce the right hand side. This is why the method of undetermined coefficients will not work.

Next consider the right hand side xe^{x^2}. Notice that when you take derivatives you get a polynomial times e^{x^2}, however the degree of the polynomial gets bigger and bigger as you take more derivatives. Therefore, the same thing will happen for this right hand side as for $(1+x^2)^{-1}$, and the method of undetermined coefficients will not work.

6. **Without solving** the following systems of differential equations, determine whether or not the origin ($u_1 = 0 = u_2$) is unstable, stable, or strictly stable.

(a) $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

We compute the eigenvalues of the coefficient matrix: λ is an eigenvalue if and only if

$$0 = \det\left(\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} = \lambda^2 - 7\lambda + 12 - 2 = (\lambda - 5)(\lambda - 2).$$

Because both eigenvalues are positive (we have $\lambda = 2$ and $\lambda = 5$), the origin is unstable.

(b) $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

We compute the eigenvalues of the coefficient matrix: λ is an eigenvalue if and only if

$$0 = \det\left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 3 \\ -1 & 1-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 2 + 3.$$

The roots of this quadratic are

$$\lambda = \frac{3 \pm i\sqrt{11}}{2}.$$

Notice that the real parts of both eigenvalues are positive, to the $u_1 = 0 = u_2$ is unstable. The solution curves spiral away from the origin.

(c) $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

We compute the eigenvalues of the coefficient matrix: λ is an eigenvalue if and only if

$$0 = \det\left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 1 + 1.$$

The roots of the quadratic are

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i\sqrt{2}.$$

Both eigenvalues have negative real parts, so $u_1 = 0 = u_2$ is strictly stable. The solution curves all spiral towards the origin.

(d) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$

We compute the eigenvalues of the coefficient matrix: λ is an eigenvalue if and only if

$$0 = \det \left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 1 - 1 = \lambda(\lambda - 2).$$

In this case the eigenvalues are $\lambda = 0$ and $\lambda = 2$. Because there is a positivve eigenvalue, the origin is unstable.

7. Find the general solution of each of the following systems.

(a) $\begin{array}{rcl} u_1 + u_2 & = & u_1' \\ -u_2 & = & u_2' \end{array}$

We can solve this first system by substitution. The second equation is separable:

$$u_2' = -u_2 \Rightarrow -x + c_2 = \int dx = \int \frac{du_2}{u_2} = \ln u_2,$$

and so $u_2(x) = c_2 e^{-x}$. Now plug this into the first equation:

$$u_1 + c_2 e^{-x} = u_1' \Rightarrow u_1' - u_1 = c_2 e^{-x}.$$

This is a first order linear ODE, so we use the integrating factor $\mu = e^{-x}$, so the equation becomes $(u_1 e^{-x})' = c_2 e^{-2x}$. Integrating, we see

$$e^{-x} u_1 = -\frac{c_2}{2} e^{-2x} + c_1 \Rightarrow u_1 = c_1 e^{x} - \frac{c_2}{2} e^{-x}.$$

(b) $\begin{array}{rcl} u_1 & = & u_2' \\ u_2 & = & -2u_1' \end{array}$

We can also solve this system bby substitution. This time put the first equation into the second:

$$u_2 = -2u_1' = -2(u_2')' = -2u_2\text{“} \Rightarrow u_2 = c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2}).$$

Then by the first equation

$$u_1 = u_2' = -\frac{c_1}{\sqrt{2}} \sin(x/\sqrt{2}) + \frac{c_2}{\sqrt{2}} \cos(x/\sqrt{2}).$$

(c) $\begin{array}{rcl} u_1 + 2u_2 & = & u_1' \\ 2u_1 + u_2 & = & u_2' \end{array}$

We will solve this system (and the next one) by finding eigenvalues and eigenvectors. The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

and so and eigenvaule λ satisfies

$$0 = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 - 4 = (\lambda - 3)(\lambda + 1) \Rightarrow \lambda = 3, -1.$$

We find the $\lambda = 3$ eigenvector:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}.$$

This forces $3a = 2b$, so we can choose

$$v_1 = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}.$$

For the $\lambda = -1$ eigenvector, we have

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix}.$$

This forces $a = b$, so we can choose

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Putting all this together, we have the general solution

$$\begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = c_1 e^{3x} \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} + c_2 e^{-x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(d) $$\begin{aligned} u_1 - u_2 &= u_1' \\ u_1 + u_2 &= u_2' \end{aligned}$$

Again we solve the system by finding eigenvalues and eigenvectors. The coefficient matrix is

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

so the eigenvalues λ satisfy

$$0 = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 2 \Rightarrow \lambda = 1 \pm i.$$

To find the $\lambda = 1+i$ eigenvector:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (1+i)a \\ (1+i)b \end{bmatrix} \Rightarrow b = -ia,$$

so we can choose the eigenvector to be

$$v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Similarly, for the $\lambda = 1-i$ eigenvalue we have

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (1-i)a \\ (1-i)b \end{bmatrix} \Rightarrow b = ia,$$

so we can choose the eigenvector to be

$$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Thus we can write the general solution as

$$\begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = c_1 e^{(1+i)x} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{(1-i)x} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

In fact, we can rewrite this general solution in terms of sines and cosines, which is more convenient for real problems. First we pull out the common factor of e^x in both terms. Then we have

$$\begin{aligned} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} &= e^x \left(c_1(\cos x - i \sin x) \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2(\cos x + i \sin x) \begin{bmatrix} 1 \\ i \end{bmatrix} \right) \\ &= e^x \begin{bmatrix} \cos x(c_1 + c_2) + i \sin x(-c_1 + c_2) \\ i \cos x(-c_1 + c_2) + \sin x(-c_1 - c_2) \end{bmatrix} \\ &= e^x \left((c_1 + c_2) \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} + i(-c_1 + c_2) \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} \right). \end{aligned}$$