

Practice Problems
Math 221
February 9, 2001

1. For each of the following differential equations, decide whether the given function is a solution.

(a) $y' = (x + 1)(y^2 - 1)$, $y = -\frac{1 + \exp(x^2 + 2x)}{1 - \exp(x^2 + 2x)}$ (here $\exp(z) = e^z$)

(b) $y' + 2xy = x$, $y = 2 - e^{x^2}$

(c) $y' = xy$, $y = e^{x^2/2}$

2. Sketch the slope field and some typical solution curves for each of the following differential equations.

(a) $y' = y^2 - 1$

(b) $y' + xy = 0$

(c) $y' = x^2 + y^2$

3. Solve each of the given initial value problems.

(a) $y' - 2xy = x$, $y(0) = 1$

(b) $y' - x^2e^{-y} = 0$, $y(0) = 4$

(c) $y' - ye^{yx} = 0$, $y(0) = 0$

(d) $y' = y^2$, $y(0) = 1$

(e) $y' + (1/x)y = x$, $y(1) = 1$

4. Consider the differential equation

$$y' = y(y^2 - 1).$$

(a) Find all the equilibria (constant solutions) of this equation.

(b) Classify each of these equilibria as unstable, stable, or strictly stable.

(c) Sketch some typical solution curves.

5. (a) Write down a differential equation with one strictly stable equilibrium. (Be sure to justify your answer.)

(b) Write down a differential equation with one stable, but not strictly stable, equilibrium. (Be sure to justify your answer.)

6. Consider the differential equation

$$y' = y(y^2 - 1) - h,$$

where h is a parameter.

(a) Does this system have a bifurcation point? If it does, find the bifurcation point.

(b) Describe how the behavior of the system depends on h .

(c) Sketch some typical solution curves, depending on the value of the parameter h .

7. Write down the differential equation which models the following phenomenon: the tangent line to the graph of any solution passes through the origin. Also, solve this differential equation with the initial conditions

$$y(1) = 1.$$

8. Consider a falling body, where the force of wind resistance is proportional to the velocity, say

$$F_R = -m\rho v,$$

where m is the mass of the body, v its velocity (in the vertical direction) and $\rho > 0$ is a constant.

(a) Write down a differential equation for v . You may assume that the only two forces acting on the body are gravity (which is a constant force of $-mg$) and wind resistance.

(b) Find the constant solutions (i.e. equilibria) and classify them as unstable, stable, or strictly stable.

(c) Sketch the slope field and some typical solution curves.

(d) Explain your picture physically. In other words, what happens to this falling body?

9. For each of the following differential equations, say whether the superposition principle holds. (Be sure to justify your answer.)

(a) $y'' + xy = x$

(b) $y'' + x^2y' - xy = 0$

(c) $y'' - y^2 = 0$

(d) $y'' + \sin(x)y = \cos(s)y'$

10. Consider a differential equation of the form

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0.$$

(a) How many linearly independent solutions does this equation have? Why?

(b) Given three solutions of this equation, how can you tell that they are linearly independent? Why does this work?

11. Consider a differential equation of the form

$$y'' + p(x)y' + q(x)y = 0,$$

and let y_1, y_2 be two solutions of this equation. Recall that the Wronskian is

$$W := y_1y_2' - y_2y_1'.$$

(a) The Wronskian W satisfies a differential equation. What is it?

(b) Conclude from this differential equation that $W = W(x)$ is either never zero or always zero.

12. Solve the following initial value problems.

(a) $y'' - 3y' + 4y = 0, y(0) = 1, y'(0) = 0$

(b) $y'' - y = 0, y(0) = -1, y'(0) = 1$

(c) $y'' + 5y' + 6y = 0, y(0) = 1, y'(0) = 1$

13. Consider the differential equation

$$y' = y^2(x - 1).$$

Note: you do not need to find the general solution to this differential equation.

(a) (2 points) Is $y = \frac{1}{1-x^2}$ a solution to this differential equation?

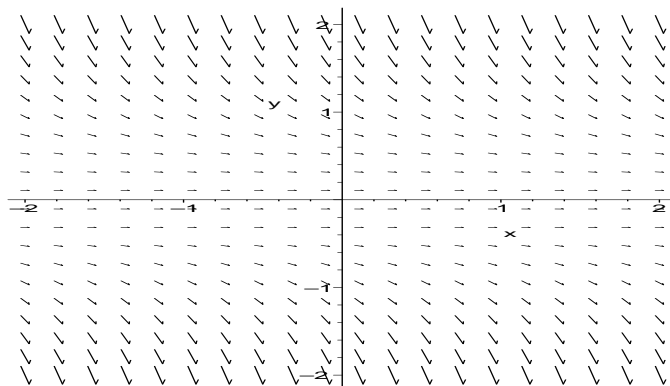
(b) (2 points) Is $y = \frac{1}{x-x^2}$ a solution to this differential equation?

14. (10 points) Solve the initial value problem

$$y' + \frac{y}{x} = x^2, \quad y(1) = 1.$$

15. Below I have drawn the slope field to a differential equation of the form

$$y' = f(x, y).$$



- (a) (3 points) Does f depend on both x and y , or only one of them? Be sure to explain your answer.
- (b) (4 points) Explain why $y = 0$ is an equilibrium solution.
- (c) (4 points) Is $y = 0$ stable?

16. Consider the differential equation

$$y' = y - y^2.$$

Note: you do not need to solve this differential equation.

- (a) (5 points) Verify that $y = 0, 1$ are the only equilibria (i.e. constant solutions) to this differential equation.
- (b) (5 points) Classify these equilibria as unstable, stable, or strictly stable.
- (c) (5 points) Sketch some typical solution curves to this differential equation.

17. Consider the differential equation

$$y'' - 3y' + 2y = 0.$$

- (a) (6 points) Find the general solution to this differential equation.
- (b) (4 points) Find the solution to the differential equation with the initial conditions

$$y(0) = 0, \quad y'(0) = 1.$$