The Fourier Transform and Related Transforms

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April 20, 2004

1 Motivation

In the text by Edwards and Penney [2], we see that one can use the Laplace transform to turn differential equations into algebraic equations, which are much easier to solve. Here, the Laplace transform of a function f is given by

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt.$$

However, we have yet to see the inverse transform of the Laplace transform. The easiest way (I know) to find the inverse Laplace transform is through the Fourier transform, which is very interesting in its own right. For this reason, I have written these notes which collect some of the elementary properties of the Fourier transform, including the Fourier inversion formula and the Laplace inversion formula.

The organization of these notes is as follows: in section 2 I discuss some basic properties of convolutions. Section 3 introduces the Fourier transform and collects some of its basic properties. Section 4 contains the Fourier inversion formula and section 5 contains some applications of the Fourier transform. Finally, section 6 discusses the inverse Laplace transform. The subject of integral transforms is vast and deep; I cannot hope to do more than scratch the surface in these notes. If you are curious and want some references, please ask me. One particularly nice reference is Folland's book([3]).

I wrote these notes more or less based on lecture notes I took from courses by Ken Bube and Gunther Uhlman, see [1] and [4].

WARNING 1: beware of floating factors of 2 and π. I will occasionally get these wrong.

WARNING 2: (far more serious) there are many computations below which I will do formally, ignoring all questions of when improper integrals converge, when one can differentiate underneath the integral sign, when one can switch two integrations, etc. These are very very important questions which every responsible mathematics student should learn about. However, right now we do not have nearly enough machinery to address these important and interesting questions. Instead, I will do the computations formally below and urge you to take a good real analysis course so that you can learn (e.g.) when it is ok to differentiate underneath the integral sign. For now, you will have to trust me that the computations below make sense on appropriate functions and that these functions are plentiful.

2 Convolutions

One can think of the convolution of two functions as an averaging of one function against the other. It is defined when f and/or g decay fast enough so that the improper integral converges. More precisely:

Definition 1 *Given two functions f and g, we define the convolution of f and g by*

$$f * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)ds.$$

For instance, if f is bounded and $\int_{-\infty}^{\infty} |g(t)|dt$ is finite then $f * g$ is well defined (i.e. the improper integral converges). In this case

$$|f * g(t)| \leq \max |f| \int_{-\infty}^{\infty} |g(s)|ds.$$

(Exercise: derive this bound.)

Suppose $f = 1$ and $\int_{-\infty}^{\infty} |g(t)|dt < \infty$. Then

$$f * g(t) = \int_{-\infty} f(t-s)g(s)ds = \int_{-\infty}^{\infty} g(s)ds$$

is the constant function $\int g$.

Suppose $f(t) = t$ and g satisfies $\int |g| < \infty$ and $\int |g'| < \infty$. Then

$$\begin{aligned} f * g'(t) &= \int_{-\infty}^{\infty} f(t-s)g'(s)ds \\ &= \int_{-\infty}^{\infty} (t-s)g'(s)ds \\ &= t\int_{-\infty}^{\infty} g(s)ds - \int_{-\infty}^{\infty} sg'(s)ds \\ &= t\int_{-\infty}^{\infty} g'(s)ds + \int_{-\infty}^{\infty} g(s)ds. \end{aligned}$$

In the last step I integrated by parts. The boundary terms in the integration by part are both zero because $g(s) \to 0$ as $s \to \pm\infty$.

Next I will list some important features of convolutions. Firstly, the convolution of two functions does not depend on the order of the functions:

$$\begin{aligned} f * g(t) &= \int_{-\infty}^{\infty} f(t-s)g(s)ds \\ &= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = g * f(t). \end{aligned}$$

Using the fact that $f * g = g * f$, one can take the derivative of $f * g$ two different ways:

$$\frac{d}{dt} f * g(t) = \frac{d}{dt} \int_{-\infty}^{\infty} f(t-s)g(s)ds = \int_{-\infty}^{\infty} \frac{d}{dt}(f(t-s)g(s))ds = \int_{-\infty}^{\infty} f'(t-s)g(s)ds$$

or

$$\frac{d}{dt} f * g(t) = \frac{d}{dt} g * f(t) = \int_{-\infty}^{\infty} g'(t-s)f(s)ds.$$

One should be a bit careful with convolutions, as they have some unusual properties. For instance, if $f(t) = \frac{1}{t+i}$ and $g(t) = \frac{1}{t-i}$ then $f * g(t) = 0$ for all t, even though neither f or g are ever zero!. The proof of this is computation involving the Fourier transform, which we may see a bit later. However, if $f * f(t) = 0$ for all t, then f is really the zero function.

Let $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$ and for $\epsilon > 0$ define $\phi_\epsilon(t) = \frac{1}{\epsilon}\phi(\frac{t}{\epsilon})$. Notice $\phi > 0$ and $\phi(t) \to 0$ as $t \to \pm\infty$. In fact,

$$\begin{aligned}(\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} dt)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2+s^2)} ds dt \\ &= \int_0^{2\pi}\int_0^{\infty} re^{-\frac{1}{2}r^2} dr d\theta \\ &= 2\pi\,(-e^{-\frac{1}{2}r^2})\Big|_{r=0}^{r=\infty} = 2\pi.\end{aligned}$$

This computation shows $\int_{-\infty}^{\infty} \phi(t)dt = 1$. By the change of variables $\tau = \frac{t}{\epsilon}$, we also have $\int_{-\infty}^{\infty} \phi_\epsilon(t)dt = \int_{-\infty}^{\infty} \phi(\tau)d\tau = 1$.

However, as $\epsilon \to 0$, $\phi_\epsilon(t)$ is very small for $|t| \geq \epsilon$. In fact, everywhere except $t = 0$, $\phi_\epsilon \to 0$ uniformly. The value of $\phi_\epsilon(0)$ grows arbitrarily large as $\epsilon \to 0$. In the limit, one obtains what is commonly called the Dirac delta mass centered at $t = 0$, denoted δ_0. You can think of δ_0 as an operator on functions defined by

$$\delta_0(f) = f(0).$$

In fact, this operation of δ_0 on the function f above is the limit of integrations of f with ϕ_ϵ. One can see this by looking at the limit of $\phi_\epsilon * f(t)$ as $\epsilon \to 0$. We can compute:

$$\int_{-\infty}^{\infty} |f(t) - \phi_\epsilon * f(t)|dt \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t) - \phi_\epsilon(t-s)f(s)|ds dt \to 0$$

as $\epsilon \to 0$. This is because for s far away from t, $\phi_\epsilon(t-s)$ is very small. This computation shows $\phi_\epsilon * f(t)$ is close to $f(t)$ for all t, because if it were not then the integral above would be bounded away from zero. Thus we have shown that as $\epsilon \to 0$, we must have $\phi_\epsilon * f \to f$.

For this reason, ϕ_ϵ is called an *approximate identity* for ϵ small. (It is also sometimes called a *mollifier*.) The reason for this terminology is that for $\epsilon > 0$ small, $\phi_\epsilon * f$ is a function which is everywhere very close to f. If f is bounded (it need not even be continuous!) then $\phi_\epsilon * f$ is a smooth functions (with infinitely many derivatives) which is everywhere close to f.

3 The Fourier Transform

One can think of the Fourier transform of a function as smearing the function values over the whole real line. More precisely:

Definition 2 *The Fourier transform of a function f is defined by*

$$\mathcal{F}(f)(s) = \hat{f}(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)dt.$$

Here i is the square root of -1.

Again, the Fourier transform $\hat{f}$ is defined only when the improper integral converges. For instance, if $\int_{-\infty}^{\infty} |f(t)|dt < \infty$, then $\hat{f}(s)$ exists for all s and

$$|\hat{f}(s)| \leq \int_{-\infty}^{\infty} |f(t)|dt.$$

In fact, under this condition ($\int |f| < \infty$), $\hat{f}(s) \to 0$ as $s \to \pm\infty$. (Exercise: derive the bound I list above and prove that in this case $\hat{f}(s) \to 0$ as $s \to \pm\infty$.)

Notice that the Fourier transform of any function f (provided f decays sufficiently so that the improper integral converges) has as many derivatives as you please: just differentiate the exponential under the integral sign. In particular, f need not even be continuous and $\hat{f}$ can still have infinitely many derivatives.

The Fourier transform enjoys many nice properties, some of which are listed below.

First note

$$\mathcal{F}(e^{i\eta t}f(t))(s) = \int_{-\infty}^{\infty} e^{-ist}e^{i\eta t}f(t)dt = \int_{-\infty}^{\infty} e^{-i(s-\eta)t}f(t)dt = \hat{f}(s-\eta).$$

Similarly,

$$\mathcal{F}(f(t-\eta))(s) = \int_{-\infty}^{\infty} e^{-ist}f(t-\eta)dt = \int_{-\infty}^{\infty} e^{-is(\tau+\eta)}f(\tau)d\tau = e^{-is\eta}\hat{f}(s).$$

Here we have used the change of variables $\tau = t - \eta$. Also, (provided f and f' decay quickly enough)

$$\mathcal{F}(f')(s) = \int_{-\infty}^{\infty} e^{-ist}f'(t)dt = e^{-ist}\Big|_{t=-\infty}^{t=\infty} + is\int_{-\infty}^{\infty} e^{-ist}f(t)dt = is\hat{f}(s).$$

Here we have used the fact that $f(t) \to 0$ as $t \to \pm\infty$ to conclude that the boundary terms in the integration by parts both go to zero. There is similarly a nice formula for $\frac{d}{ds}\hat{f}(s)$:

$$\frac{d}{ds}\hat{f}(s) = \frac{d}{ds}\int_{-\infty}^{\infty} e^{-ist}f(t)dt = \int_{-\infty}^{\infty} \frac{d}{ds}e^{-ist}f(t)dt = \int_{-\infty}^{\infty} e^{-ist}(-itf(t))dt = \mathcal{F}(-itf).$$

These properties (and indeed their derivations) are very similar some of the properties of the Laplace transform derived in the text [2].

The Fourier transform of a convolution has a very pretty expression:

$$\begin{aligned}
\mathcal{F}(f*g)(s) &= \int_{-\infty}^{\infty} e^{-ist}f*g(t)dt \\
&= \int_{-\infty}^{\infty} e^{-ist}\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ist}f(t-\tau)g(\tau)dtd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ist}e^{-is\tau}e^{is\tau}f(t-\tau)g(\tau)dtd\tau \\
&= \int_{-\infty}^{\infty} e^{-is(t-\tau)}f(t-\tau)d(t-\tau)\cdot\int_{-\infty}^{\infty} e^{-is\tau}g(\tau)d\tau \\
&= \hat{f}(s)\cdot\hat{g}(s).
\end{aligned}$$

Thus the Fourier transform of a convolution is the product of the Fourier transforms.

There is one Fourier transform we can compute rather quickly. Let $f(t)$ be defined by

$$f(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & t < 0. \end{cases}$$

Then

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-ist}f(t)dt = \int_{0}^{\infty} e^{-t(1+-is)}ds = -\frac{1}{1+is}\Big|_{t=0}^{t=\infty} = \frac{1}{1+is}.$$

The reason we truncated the exponential in f was to ensure that f has sufficient decay so that the Fourier transform is well-defined. Recall that $e^{-t} \to \infty$ as $t \to -\infty$, so if we let $g(t) = e^{-t}$ then $\int_{-\infty}^{\infty} e^{-ist}g(t)dt = \int_{-\infty}^{\infty} e^{-ist}e^{-t}dt$ is a divergent improper integral.

We can also compute the Fourier transform of $tf(t)$:

$$\mathcal{F}(tf)(s) = i\frac{d}{ds}\hat{f}(s) = i\frac{d}{ds}(\frac{1}{1+is}) = \frac{1}{(1+is)^2}.$$

Again, to do this computation rigorously, one must justify that $tf(t)$ decays fast enough so that its Fourier transform is well defined.

Recall that we defined $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$ in section 2. To find the Fourier inversion formula we will need the following lemma.

Lemma 1 $\hat{\phi} = \phi$.

Proof: Notice $\phi'(t) = -t\phi(t)$. Combining this with $\frac{d}{ds}\hat{\phi} = \mathcal{F}(-it\phi)$ implies $is\hat{\phi} = -i\frac{d}{ds}\hat{\phi}$. We can rewrite this equation as

$$\frac{d}{ds}\hat{\phi} + s\hat{\phi} = 0.$$

This is an ODE in s, and it has solutions $\hat{\phi}(s) = \hat{\phi}(0)e^{-\frac{1}{2}s^2}$. All that remains is to find $\hat{\phi}(0)$. This is

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}dt = 1.$$

□

Notice that by the change of variables $\tau = \frac{t}{\epsilon}$, the same proof shows $\hat{\phi}_\epsilon = \phi_\epsilon$. The above lemma shows

$$\phi_\epsilon(t) = \frac{1}{\epsilon}\phi(\frac{t}{\epsilon}) = \frac{1}{\epsilon}\hat{\phi}(\frac{t}{\epsilon}) = \frac{1}{\epsilon}\int_{-\infty}^{\infty} e^{-i\frac{t}{\epsilon}s}\phi(s)ds = \int_{-\infty}^{\infty} e^{-it\tau}\phi(\epsilon\tau)d\tau = \int_{-\infty}^{\infty} e^{it\eta}\phi(\epsilon\eta)d\eta.$$

Here we have used two changes of variables: first $\tau = \frac{s}{\epsilon}$ and then $\eta = -\tau$. We have also used $\phi(\eta) = \phi(-\eta)$ (i.e., ϕ is an even function). This formula $\phi_\epsilon(t) = \int_{-\infty}^{\infty} e^{it\eta}\phi(\epsilon\eta)d\eta$ is the formula we will use to derive the Fourier inversion formula.

4 The Fourier Inversion Formula

In this section we will derive a formula for the inverse Fourier transform $\mathcal{F}^{-1}(f) = \check{f}(t)$.

Theorem 2 $f(t) = \int_{-\infty}^{\infty} e^{ist}\hat{f}(s)ds$.

Before embarking upon the proof of this beautiful theorem, I am obligated to remind you that there are many many computations in the proof below which require more justification than I will give. The full justification of this theorem is rather complicated and far beyond the scope of this course. However, the formal computation is not too bad. (Exercise: find all the places in the proof below where you need to justify a formal computation.)

Proof: Recall that $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t}$ and $\phi_\epsilon(t) = \frac{1}{\epsilon}\phi(\frac{t}{\epsilon})$ for $\epsilon > 0$. As I mentioned in the last section, the important fact we will use is

$$\phi_\epsilon(t) = \int_{-\infty}^{\infty} e^{it\eta}\phi(\epsilon\eta)d\eta.$$

Then

$$\begin{aligned}
f*\phi_\epsilon(t) &= \int_{-\infty}^{\infty} f(t-s)\phi_\epsilon(s)ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-s)e^{is\eta}\phi(\epsilon\eta)d\eta ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-s)e^{is\eta}e^{-it\eta}e^{it\eta}\phi(\epsilon\eta)d\eta ds \\
&= \int_{-\infty}^{\infty} f(t-s)e^{-i\eta(t-s)}e^{it\eta}\phi(\epsilon\eta)d\eta ds \\
&= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} e^{-i\eta(t-s)}f(t-s)d(t-s))e^{it\eta}\phi(\epsilon\eta)d\eta \\
&= \int_{-\infty}^{\infty} \hat{f}(\eta)e^{it\eta}\phi(\epsilon\eta)d\eta \\
&\to \int_{-\infty}^{\infty} e^{it\eta}\hat{f}(\eta)d\eta
\end{aligned}$$

as $\epsilon \to 0$. However, $f*\phi_\epsilon(t) \to f(t)$ as $\epsilon \to 0$, which completes the proof of the theorem. □

This yields a formula for the inverse transform of the Fourier transform:

$$\mathcal{F}^{-1}(g)(t) = \check{g}(t) = \int_{-\infty}^{\infty} e^{ist}g(s)ds.$$

The above formula is know as the Fourier inversion formula.

You may recall that back in section 2 I listed two nowhere zero functions $f(t) = \frac{1}{t+i}$ and $g(t) = \frac{1}{t-i}$ such that the convolution $f*g$ is everywhere zero. Now we can finally prove this remarkable property. If we define h by

$$h(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & t < 0 \end{cases}$$

we have already seen that $\hat{h}(s) = -i\frac{1}{s-i}$. Also, $\hat{h}(-s) = i\frac{1}{s+i}$. This implies

$$f(s) = \frac{1}{s-i} = i\hat{h}(s) \qquad g(s) = \frac{1}{s+i} = -i\hat{h}(-s).$$

Now we are ready to compute the convolution.

$$f*g(t) = (\mathcal{F}(f(s)) * \mathcal{F}(f(-s)))(t) = \mathcal{F}(f(s)\cdot f(-s)) = 0$$

because for all s either $f(s) = 0$ or $f(-s) = 0$ (and the Fourier transform of zero is zero).

5 Some Applications of the Fourier Transform

The Fourier transform has many applications in many different fields. Indeed, one can view modern signal processing as an extended application of the Fourier transform. In this section I will just mention a couple of these applications.

5.1 Convection Equation

Suppose you again have a thin rod of which conducts heat uniformly. Then the heat content of the rod at distance x and time t is given by $u(x,t)$ where

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + bu,$$

where a and b are constants. We will need some initial conditions, which we write as $u(x,0) = f(x)$. You can think of $f(x)$ as the initial heat content of the rod.

To solve this differential equation we will take the Fourier transform of both sides of the equation in the x variable (but NOT the t variable). Let ξ be the dual variable to x. Then upon taking the Fourier transform we find

$$\int_{-\infty}^{\infty} e^{ix\xi}\partial_t u(x,t)dx = \int_{-\infty}^{\infty} e^{-ix\xi}(a\partial_x u(x,t) + bu(x,t))dx.$$

We will first analyze the left hand side of this equation:

$$\int_{-\infty}^{\infty} e^{-ix\xi}\partial_t u(x,t)dx = \partial_t \int_{-\infty}^{\infty} e^{-ix\xi} u(x,t)dx = \partial_t \hat{u}(\xi,t).$$

Here we have used the fact that the integrand does not depend on t, so we can pull the differentiation with respect to t out of the integral.

Next we analyze the right hand side of the equation:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ix\xi}(a\partial_x u(x,t) &+ bu(x,t))dx \\ &= a(e^{-ix\xi}u(x,t)\big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty}(-i\xi)e^{-ix\xi}u(x,t)dx) + b\int_{-\infty}^{\infty} e^{-ix\xi}u(x,t)dx \\ &= a(i\xi\int_{-\infty}^{\infty} e^{-ix\xi}u(x,t)dx) + b\int_{-\infty}^{\infty} e^{-ix\xi}u(x,t)dx = (ia\xi + b)\hat{u}(\xi,t).\end{aligned}$$

In this computation we integrated by parts to get rid of the differentiation with respect to x inside the integral. Then we used the fact that u has to decay (i.e. $u(x,t) \to 0$ as $x \to \pm\infty$) in order for the Fourier transform to make sense. This fact eliminates the boundary terms in the integration by parts.

At any rate, after taking the Fourier transform we obtain the following equation:

$$\partial_t \hat{u}(\xi,t) = (ia\xi + b)\hat{u}(\xi,t).$$

This is now a one parameter family of ODEs in the variable t, parameterized by ξ. In other words, for each fixed ξ we get an ODE in the variable t, and the ODE changes continuously as we change ξ continuously. In fact, this is a nice separable first order ODE. We can solve it by first dividing by $\hat{u}$ and then integrating both sides with respect to t:

$$(ia\xi + b)t + c = \int (ia\xi + b)dt = \int \frac{1}{\hat{u}}\frac{\partial \hat{u}}{\partial t}dt = \int \frac{d\hat{u}}{\hat{u}} = \log \hat{u}.$$

Taking the exponential of both side of the equation $(ia\xi + b)t + c = \log \hat{u}$ we obtain

$$\hat{u}(\xi,t) = e^{(ia\xi+b)t+c} = \hat{c}(\xi)e^{(ia\xi+b)t}.$$

But this constant of integration $\hat{c}$ is just $\hat{u}(\xi,0) = \hat{f}(\xi)$, so we can write our solution as

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{(ia\xi+b)t}.$$

Now we are almost done. We have found $\hat{u}(\xi,t)$, but we are really interested in finding $u(x,t)$. To find u, we will take the inverse Fourier transform. Fortunately, we do not really need to evaluate any integrals to take the Fourier transform, as we will see below. Again we start with the left hand side of the equation:

$$\mathcal{F}^{-1}(\hat{u})(x,t) = \int_{-\infty}^{\infty} e^{ix\xi}\hat{u}(\xi,t)d\xi = u(x,t)$$

by the Fourier inversion formula. Taking the inverse Fourier transform of the right hand side of the equation we obtain

$$\begin{aligned}\mathcal{F}^{-1}(\hat{f}e^{(ia\xi+b)t})(x,t) &= \int_{-\infty}^{\infty} e^{ix\xi}e^{(ia\xi+b)t}\hat{f}(\xi)d\xi = \int_{-\infty}^{\infty} e^{bt}e^{ix\xi+ia\xi}\hat{f}(\xi)d\xi \\ &= e^{bt}\int_{-\infty}^{\infty} e^{i\xi(x+at)}\hat{f}(\xi)d\xi = e^{bt}f(x+at).\end{aligned}$$

Thus the solution we seek is

$$u(x,t) = e^{bt}f(x+at).$$

Notice that $u(x,t)$ grows exponentially in t, with a rate of growth determined by b. This is a phenomenon called *convection*.

5.2 The wave equation

Suppose you have a vibrating string stretched taut. If $u(x,t)$ is the vertical displacement of the string at time t and horizontal position x then

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2}.$$

Again, we will need an initial condition, $u(x,0) = f(x)$. You can think of this initial condition as giving the initial position of the string. This is the wave equation for a traveling wave; the constant a is called the wave speed.

We will find solutions to the wave equation using two different methods below. First we will factor the equation and reduce it to a special case of the convection equation. Then we will solve it without factoring.

We can factor this equation as follows:

$$(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2})u = (\frac{\partial}{\partial t} - a\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})u = 0.$$

We could equivalently write the wave equation as

$$(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x})u = 0.$$

(Exercise: Check that $L_- = \frac{\partial}{\partial t} - a\frac{\partial}{\partial x}$ and $L_+ = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$ commute, i.e. $L_-L_+(u) = L_+L_-(u)$. It is crucial in this computation that a is constant.)

The point of writing the wave equation in this fashion is that from this factorization one can conclude that u is a solution to the wave equation (i.e. $\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$) if and only if either $\frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = 0$ or $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$. Thus it suffices to find solutions to the equations

$$\partial_t u = a\partial_x u \qquad \partial_t u = -a\partial_x u.$$

You may recognize these two equations as special cases of the convection equation, corresponding to $b = 0$. Indeed, we can solve these equations by taking the Fourier transform (in the variable x) of both sides of the equation, obtaining

$$\partial_t\hat{u}(\xi,t) = \pm ia\xi\hat{u}(\xi,t).$$

This is a one parameter family of ODEs in the variable t, parameterized by ξ. It is separable, and solutions are given by

$$\hat{u}(\xi,t) = e^{\pm ia\xi t}\hat{u}(\xi,0) = e^{\pm ia\xi t}\hat{f}(\xi).$$

See the previous subsection for more details about this computation. Taking the inverse Fourier transform, we obtain the solution

$$u(x,t) = f(x \pm at).$$

Notice we can write the general solution of the wave equation as a superposition:

$$u(x,t) = c_+ f(x+at) + c_- f(x-at).$$

The term with $f(x+at)$ essentially translates the initial wave shape to the left; it is called the left traveling wave. Similarly, the term with $f(x-at)$ translates the initial wave shape to the right and is called the right moving wave. Thus we have written the general solution to the wave equation as the superposition of a left traveling wave and a right traveling wave. This should make some physical sense. If you puck a string that is stretched taut then you should be able to see the motion of that string as the sum of an impulse moving to the right and another impulse moving to the left. If you want to get a better feel for how solutions to the wave equation behave, you can try this at home.

A second way to solve the wave equation is to just take the Fourier transform (again, in the x variable) of both sides of the equation right away. Then we obtain

$$\partial_t^2 \hat{u}(\xi,t) = a^2(-\xi^2)\hat{u}(\xi,t).$$

This is again a one parameter family of ODEs in the variable t, parameterized by ξ. However, this time it is a linear second order ODE. Fortunately, we know how to solve this ODE as well. If we write it as

$$\hat{u}''(\xi,t) + a^2\xi^2\hat{u}(\xi,t) = 0$$

(where $'$ denotes taking a derivative with respect to t) we see that this is linear with constant (in t) coefficients with characteristic polynomial $r^2 + a^2\xi^2 = 0$. The roots of this characteristic polynomial are $r = \pm ia\xi$, which means a basis for the solution space is

$$\hat{u}_+(\xi,t) = e^{ia\xi t}\hat{u}(\xi,0) = e^{ia\xi t}\hat{f}(\xi) \qquad u_-(\xi,t) = e^{-ia\xi t}\hat{u}(\xi,0) = e^{-ia\xi t}\hat{f}(\xi).$$

Notice this method yields precisely the same solutions as before. (For any given differential equation there is usually a plethora of equally valid ways to solve it.) As before, we obtain solutions as a superposition of left and right traveling waves:

$$u(x,t) = c_+ f(x+at) + c_- f(x-at).$$

5.3 The Heat Equation

Suppose you have a thin rod which conducts heat uniformly. If the temperature at the ends of the rod is heat constant, then the temperature distribution $u(x,t)$ obeys the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

We will also have to set some initial condition, which you can think of as the initial temperature of the rod: $u(x,0) = f(x)$.

To solve this equation, we will first take the Fourier transform of both sides in the x variable. This yields

$$-\xi^2\hat{u}(\xi,t) = \frac{\partial \hat{u}}{\partial t}(\xi,t),$$

which is now a family of ordinary differential equations in t for $\hat{u}$, parameterized by ξ. (Here ξ is the dual variable to x.) For each ξ, we have the initial condition $\hat{u}(\xi,0) = \hat{f}(\xi)$. So the initial value problem has the solution

$$\hat{u}(\xi,t) = e^{-t\xi^2}\hat{f}(\xi).$$

Now to find the solution u to our original equation, we use the Fourier inversion formula:

$$u(x,t) = \int_{-\infty}^{\infty} e^{ix\xi} \hat{u}(\xi, t) d\xi = \int_{-\infty}^{\infty} e^{ix\xi} e^{-t\xi^2} \hat{f}(\xi) d\xi.$$

One can used this formula for the solution u to show

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \leq \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

In fact, if you look at the behavior of the solution u with

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1, \end{cases}$$

you will find that the solution $u(x,t)$ spreads out more and more as t becomes larger and larger. The phenomenon is called *dissipation*.

6 The Inverse Laplace Transform

It turns out that the key thing to do here is to rewrite the Laplace transform in terms of the Fourier transform and then use the Fourier inversion formula.

Recall that the Laplace transform of $f(t)$ is given by

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

If we let $s = \sigma + i\tau$ then we can rewrite the Laplace transform as

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-\sigma t} e^{-i\tau t} f(t) dt = \mathcal{F}(e^{-\sigma t} f)(\tau) = \mathcal{F}(g_\sigma)(\tau)$$

where $g_\sigma(t) = e^{-\sigma t} f(t)$. Then by the Fourier inversion formula we have

$$g_\sigma(t) = \mathcal{F}^{-1}(\mathcal{L}(f))(t) = \int_{-\infty}^{\infty} e^{i\tau t} \mathcal{L}(f)(s) d\tau.$$

Now we substitute back in that $g_\sigma(t) = e^{-\sigma t} f(t)$ and $s = \sigma + i\tau$ to see that

$$f(t) = e^{\sigma t} \int_{-\infty}^{\infty} e^{it\tau} \mathcal{L}(f)(s) d\tau = \int_{-\infty}^{\infty} e^{(\sigma + i\tau)t} \mathcal{L}(s) d\tau = \int_{-\infty}^{\infty} e^{ist} \mathcal{L}(s) d\tau.$$

Notice now that $\sigma = \Re s$ is a parameter in the inversion formula for $\mathcal{L}$. Changing this parameter will change the exponential rate of growth of the function the inverse transform produces.

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