Solutions to the Midterm Exam

Math 221

March 31, 2007

1. Consider the following differential equation:

$$u'' - 3u' + 2u = e^x$$

   (a) (3 points) Write down the associated homogeneous equation and find it's general solution.
   The homogeneous equation is

$$u'' - 3u' + 2u = 0.$$

   To find solutions, we look for something of the form $u(x) = e^{rx}$, and so

$$0 = r^2 - 3r + 2 = (r-2)(r-1) \Rightarrow r = 1, 2,$$

   which implies the general solution is

$$u = c_1 e^x + c_2 e^{2x}.$$

   (b) (4 points) Find a particular solution to the inhomogeneous equation.
   The right hand side is a solution to the homogeneous, so we start by guessing something that's the homogeneous solution multiplied bby x: $u_p(x) = Axe^x$ for some constant A. Then

$$u_p' = Axe^x + Ae^x, \qquad u_p'' = Axe^x + 2Ae^x.$$

   Plugging this in, we have

$$e^x = u_p'' - 3u_p' + 2u_p = Axe^x + 2Ae^x - 3(Axe^x + Ae^x) + 2Axe^x = -Ae^x \Rightarrow A = -1,$$

   and so a particular solution to the inhomogeneous equation is $u_p = -xe^x$.

   (c) (3 points) Write down the general solution to the inhomogeneous equation and solve the initial value problem with $u(0) = 1$ and $u'(0) = -1$.
   The general solution is the sum of the particular solution we found and the general solution to the homogeneous equation:

$$u = -xe^x + c_1 e^x + c_2 e^{2x}.$$

   Now we plug in the initial conditions to find c_1 and c_2:

$$1 = u(0) = c_1 + c_2, \qquad -1 = u'(0) = -1 + c_1 + 2c_2.$$

   Subtracting these two equations, we have $-c_2 = 1 \Rightarrow c_2 = -1$. Then plugging this into either equation we see $c_1 = 2$. Thus the solution to this initial value problem is

$$u(x) = -xe^x + 2e^x - e^{2x}.$$

2. Consider a forced spring system (neglect damping)

$$mx'' + kx = F\cos(\omega t).$$

   (a) (4 points) Explain why the natural (i.e. unforced) frequency of the spring is $\omega_0 = \sqrt{k/m}$.
   The unforced equation has $F = 0$, so it is

$$mx'' + kx = 0.$$

   The solutions to this equation are

$$x(t) = c_1 \cos(t\sqrt{k/m}) + c_2 \sin(t\sqrt{k/m}),$$

   and oscillate with the frequency $\omega_0 = \sqrt{k/m}$.

   (b) (4 points) When $\omega \neq \omega_0$, find a particular solution for the forced spring system.
   We try a particular solution of the form

$$x_p = A\cos\omega t + B\sin\omega t \Rightarrow x_p' = -A\omega\sin\omega t + B\omega cos\omega t, \quad x_p'' = -\omega^2(A\cos\omega t + B\sin\omega t).$$

Plugging this into the equation, we have

$$F\cos\omega t = -m\omega^2(A\cos\omega t + \sin\omega t) + k(A\cos\omega t + B\sin\omega t) = (k - m\omega^2)A\cos\omega t + (k - m\omega^2)B\sin\omega t,$$

which implies

$$A = \frac{F}{k - m\omega^2}, \qquad B = 0.$$

In other words, a particular solution is

$$x_p(t) = \frac{F}{k - m\omega^2}\cos\omega t.$$

(c) (4 points) Solve the initial value problem $x(0) = 0 = x'(0)$ for the forced spring system. Again, assume $\omega \neq \omega_0$.

Again, the general solution is the sum of the particular solution we've found and the general solution to the homogeneous equation, which is

$$x = \frac{F}{k - m\omega^2}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

Plugging in the initial conditions, we have

$$0 = x(0) = \frac{F}{k - m\omega^2} + c_1, \qquad 0 = x'(0) = -\omega_0 c_2 \Rightarrow c_1 = -\frac{F}{k - m\omega^2}, c_2 = 0$$

and the solution is

$$x = \frac{F}{k - m\omega^2}(\cos\omega t - \cos\omega_0 t).$$

(d) (4 points) Find the period of the beats for the solution of the initial value problem you just found. (Hint: you should be able to write the solution you just found as $A\sin(c_1 t)\sin(c_2 t)$ for an appropriate choice of c_1 and c_2. The trig identity $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ might be helpful.)

We start by letting

$$\alpha := \frac{\omega + \omega_0}{2}, \quad \beta = \frac{\omega - \omega_0}{2} \Rightarrow \omega = \alpha + \beta, \quad \omega_0 = \alpha - \beta.$$

Then we can rewrite the solution we just found as

$$\begin{aligned} x(t) &= \frac{F}{k - m\omega^2}(\cos(\alpha t + \beta t) - \cos(\alpha t - \beta t)) \\ &= \frac{F}{k - m\omega^2}(\cos\alpha t\cos\beta t - \sin\alpha t\sin\beta t - \cos\alpha t\cos\beta t - \sin\alpha t\sin\beta t) \\ &= -\frac{2F}{k - m\omega^2}\sin\alpha t\sin\beta t = -\frac{2F}{k - m\omega^2}\sin((t/2)(\omega + \omega_0))\sin((t/2)(\omega - \omega_0)). \end{aligned}$$

Now let's look at this form of the solution. It's the product of a high frequency oscillation, the $\sin((t/2)(\omega + \omega_0))$ term, and a low frequency oscillation, the $\sin((t/2)(\omega - \omega_0))$ term. One can think of the low frequency oscillation as a time-changing amplitude for the high frequency oscillation, and so the frequency of the beats is $|\omega - \omega_0|/2$, which means the period is

$$T = \frac{2\pi}{|\omega - \omega_0|/2} = \frac{4\pi}{|\omega - \omega_0|}.$$

3. (4 points) Consider the system

$$A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'.$$

Is the solution $u_1 \equiv 0, u_2 \equiv 0$ at the origin unstable, stable, or strictly stable? Be sure to justify your answer.

The eigenvalues λ of the coefficient matrix A satisfy

$$0 = \det\begin{bmatrix} -4 - \lambda & 3 \\ -3 & 1 - \lambda \end{bmatrix} = \lambda^2 + 3\lambda + 5 \Rightarrow \lambda = \frac{-3 \pm \sqrt{9 - 20}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{11}}{2}.$$

Notice that both eigenvalues have negative real parts, which means that all solutions decay to 0, and so the origin is strictly stable.

4. Consider the system

$$A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'$$

(a) (4 points) Verify that the eigenvalues of the coefficient matrix A are $\lambda = -3$ and $\lambda = 3$.
The eigenvalues λ of the coefficient matrix satisfy

$$0 = \det\begin{bmatrix} 2-\lambda & 5 \\ 1 & -2-\lambda \end{bmatrix} = \lambda^2 - 9 = (\lambda+3)(\lambda-3) \Rightarrow \lambda = 3, -3.$$

(b) (4 points) Is the solution $u_1 \equiv 0, u_2 \equiv 0$ at the origin unstable, stable, or strictly stable? Be sure to justify your answer.
Notice that one of the eigenvalues is positive, which means that the origin is unstable.

(c) (4 points) Find the eigenvectors of A and write down the general solution of the system.
First we find the $\lambda = 3$ eigenvector:

$$\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix} \Rightarrow a = 5b,$$

and so we can choose the eigenvector to be

$$\vec{v_1} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Next we find the $\lambda = -3$ eigenvector:

$$\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3a \\ -3b \end{bmatrix} \Rightarrow a = -b,$$

and so we can choose the eigenvector to be

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Note: you can multiply either of these eigenvectors by a constant without changing the final answer. Putting this all together, we see that the general solution is

$$\begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = c_1 e^{3x}\begin{bmatrix} 5 \\ 1 \end{bmatrix} + c_2 e^{-3x}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(d) (4 points) Solve the initial value problem with $u_1(0) = 2$ and $u_2(0) = -1$. What happens to this solution as $x \to \infty$?
We match the initial conditions:

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 5c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix}.$$

This gives us the 2×2 system of equations

$$1 = 5c_1 - c_2, \qquad -1 = c_1 + c_2 \Rightarrow c_1 = \frac{1}{6}, c_2 = -\frac{7}{6},$$

and so the solution to the initial value problem is

$$\begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \frac{1}{6}e^{3x}\begin{bmatrix} 5 \\ 1 \end{bmatrix} - \frac{7}{6}e^{-3x}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Notice that when $x \to \infty$ the second term becomes very small. This means the solution points nearly in the $(5,1)$ direction, or in other words $\lim_{x\to\infty}(u_1(x)/u_2(x)) = 5$.

(e) (4 points) For a general initial condition, what happens to the solution when $x \to \infty$?
Look at the form of the general solution we wrote earlier:

$$\begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = c_1 e^{3x}\begin{bmatrix} 5 \\ 1 \end{bmatrix} + c_2 e^{-3x}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Again, as $x \to \infty$ the second term becomes small, and so the solution points nearly in the $(5,1)$ direction. Another way to say this is to say

$$\lim_{x\to\infty}\frac{u_1(x)}{u_2(x)} = 5.$$

The exception to this is the case where the initial condition lies on the line spanned by $(-1,1)$. In this case, the coefficient c_1 above will be zero, and so the solution decays to zero.